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AN INTERSECTION HOMOLOGY INVARIANT FOR KNOTS IN A RATIONAL HOMOLOGY 3-SPHERE

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THE purpose of this paper is to define a family of computable homological invariants of knots that generalize Casson's invariant of knots. Let K be a homologically trivial knot in a rational homology 3-sphere N. Given a pair of integers (n, d) with $n \geq 1$, we define a numerical invariant $\lambda_{n,d}$ and a related polynomial invariant $p_{n,d}(t)$ which depend only on $(N, K, n, d \bmod n)$. The invariant $\lambda_{n,d}$ can be thought of as an algebraic count of the number of characters of representations of the fundamental group of the complement of K into the Lie group $SU(n)$ which take a longitude to $e^{2\pi i d/n}$ times the identity. The case where n and d are not relatively prime is of most interest to us here as the relatively prime case (for fibered knots) has been treated in [4].

While we define the invariants $\lambda_{n,d}$ and $p_{n,d}(t)$ for *any* homologically trivial knot K in a rational homology sphere N, they are most readily computed when K is a fibered knot. In this case we show that $\lambda_{n,d}$ and the coefficients of $p_{n,d}(t)$ can be computed by evaluating certain polynomials (in many variables), depending only on $(n, d \bmod n)$, at a set of numbers obtained by evaluating the (normalized) Alexander polynomial of K and its derivatives of order less than or equal to $2n - 2$ at various m-th roots of unity, $m \leq n$; see Theorems 5.21 and 5.22. Furthermore, an algorithm is given for determining these polynomials. We explicitly evaluate $\lambda_{2,0}$; see Theorem 6.4. For fibered knots, $\lambda_{n,d}$ can be computed from the intersection homology Lefschetz number of the monodromy action on the moduli space of semistable holomorphic bundles of rank n and degree d and fixed determinant over a compact Riemann surface. For n and d *not* relatively prime, this moduli space is typically singular. Our computation relies heavily on the theory developed by Frances Kirwan ([14, 15, 16, 17]) for desingularizing these spaces and computing their (mid-perversity) intersection homology.

The polynomial invariants which we define in §3 are best understood in the abstract framework of "cobordism functors" developed in §1. The axioms for such functors are somewhat reminiscent of, albeit less restrictive than, the axioms proposed for so-called "topological quantum field theories" (see [1]). In [6] it was shown how the Alexander polynomial arises in an elementary fashion from $U(1)$ representations in the context of cobordism functors (see [6, Theorem 4.4]). Here, this is generalized to $PU(n)$ representations from which we can obtain polynomial invariants which, at least in the case of fibered knots, are computable in terms of data derived from the Alexander polynomial.

In order to put our theory into perspective, we first review the definition of Casson's invariant. Let M be an oriented homology 3-sphere. Let H_1 and H_2 be two handlebodies so

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that the union of H_1 and H_2 is M and the intersection of H_1 and H_2 is a surface F. Let $R(Z)$ denote the space of representations of the fundamental group of the space Z into $SU(2)$ endowed with the topology of pointwise convergence. There is an action of $SU(2)$ on the space $R(Z)$ by conjugation. Let $X(Z)$ denote the quotient space. Let $X^i(Z)$ denote those conjugacy classes that come from irreducible representations. The following pushout diagram of fundamental groups:

$$\begin{tikzcd}
 & \pi_1(H_1) \arrow[dr] & \\
\pi_1(F) \arrow[ur] \arrow[dr] & & \pi_1(M) \\
 & \pi_1(H_2) \arrow[ur] &
\end{tikzcd}$$

gives rise to a diagram of spaces:

$$\begin{tikzcd}
 & X^i(H_1) \arrow[dl] & \\
X^i(F) & & X^i(M) \arrow[ul] \arrow[dl] \\
 & X^i(H_2) \arrow[ul] &
\end{tikzcd}$$

The space $X^i(F)$ is an orientable $6g - 6$ manifold where the genus of F is g. A choice of the orientation of F determines an orientation of $X^i(F)$. The spaces $X^i(H_j)$ are orientable $3g - 3$ manifolds. Although there is no canonical orientation for these spaces an orientation of one determines an orientation of the other. We denote the spaces $X^i(H_j)$ with a choice of orientation as Q_j. It is an easy consequence of the fact that M is a homology sphere that the intersection of the spaces Q_1 and Q_2 is compact. Hence it is possible to make them transverse by means of a compactly supported perturbation. The resulting intersection number, when correctly normalized, is Casson's invariant of M, denoted $\lambda(M)$.

In order to calculate $\lambda(M)$, Casson defines an invariant of knots in homology spheres. The knot invariant is a homological invariant, hence it can be easily computed. Even though Casson's invariant is not homological, it too can be readily computed using the knot invariant. An example of a similar phenomenon is given by the signature of a knot. The signature of a knot is not determined by the Alexander polynomial of the knot; however, it can be computed by passing through a skein tree for the knot and keeping track of the value of the Alexander polynomials at -1. Casson's invariant for knots is defined as follows. Let K be an oriented knot in M. Let $M(K, 1/n)$ denote the result of doing $1/n$ surgery on the knot K. Define $\lambda'(K)$ to be $\lambda(M(K, 1/(n+1)) - \lambda(M(K, 1/n))$. The invariant $\lambda'(K)$ can be computed from the normalized Alexander polynomial of the knot, in fact, $\lambda'(K) = \frac{1}{2}\Delta_K''(1)$.

The reason that Casson's invariant for knots can be computed in terms of the Alexander polynomial is that it can be defined as the intersection number of two canonically defined cycles in a compact space of representations related to a Seifert surface of the knot. The point seems to be that the intersection theory of such a space of representations can be computed inside a direct summand of the cohomology of the classifying space of the gauge group of a bundle over that surface. The homology theory of such a space can be seen to factor through the homology theory of the Seifert surface. Hence nothing deeper than homological data can be derived from such an approach. This has the disadvantage that homological invariants are presumably far from sufficient for understanding the topology of 3-manifolds; it has the advantage that invariants of 3-manifolds similar to Casson's invariant can be rendered computable via such a process.

Casson's invariant of manifolds can be extended to give invariants of 3-manifolds coming from representations into $SU(n)$. There are a number of approaches to such an extension; for example, one method can be derived from the techniques of [3]. It is clear that in order to compute such invariants, Casson's invariant of knots must be extended to $SU(n)$. This has already been achieved partially in the papers [4] and [5]. There, a family of knot invariants is investigated which are derived from the study of representations of knots that send the longitude of the knot to a primitive element of the center of $SU(n)$. These invariants can be easily defined as an intersection number because the corresponding spaces of representations are manifolds and the cycles are submanifolds. In this paper we extend these invariants to the case where the longitude of the knot is sent to an element of the center of $SU(n)$ that is not primitive. The problem is that the spaces involved are not manifolds. Since the spaces involved are stratified in a nice way, it is natural to attempt to use intersection homology to define intersection numbers. In the case of a fibered knot, this is relatively straightforward as the intersection number of interest can be defined using the mid-perversity intersection homology Lefschetz number of the monodromy action. The general case requires the more elaborate theory of §2 which takes into account the natural (stratum dependent) perversities of the relevant correspondences in the singular varieties in which they reside.

§1. PROJECTIVE GRADED COBORDISM FUNCTORS

In this section we define the notion of a *projective graded cobordism functor*; see Definition 1.1. Interesting examples of such functors will be constructed in §3. Given a projective graded cobordism functor together an integer k, one can associate a formal power series to any pair (M, ζ) consisting of a closed oriented 3-manifold M and a nonzero, primitive homology class $\zeta \in H_2(M)$; see Definition 1.2. Theorem 1.3 asserts that this series is independent of the various choices involved in its definition, up to multiplication by a unit of a certain type. Our formalism gives rise to knot invariants as follows. Let $K \subset N$ be a homologically trivial knot inside a closed oriented 3-manifold N. Let M be the closed oriented 3-manifold obtained by longitudinal surgery on K, and let $\zeta \in H_2(M)$ be the homology classed represented by a "capped off" Seifert surface for K. Then the series associated to the (M, ζ) via a projective graded cobordism functor and the choice of an integer k is an invariant for the knot K.

For each integer $g \geq 0$ choose a closed connected oriented surface Σ_g of genus g together with a basepoint. These surfaces will serve as models. It will also be convenient to let Σ_{-1} denote the empty surface. If S is an oriented surface a *marking* is an orientation preserving homeomorphism $\phi\colon \Sigma_g \to S$ for some g. A *marked surface* $\mathbf{S}$ consists of a j-tuple, for some positive integer j, of connected closed oriented surfaces $(S_1, \ldots, S_j)$ together with a j-tuple of markings $(\phi_1, \ldots, \phi_j)$ where $\phi_i\colon \Sigma_{g(i)} \to S_i$. We refer to the j-tuple $(\Sigma_1, \ldots, \Sigma_j)$ as the *domain* of the marked surface $\mathbf{S}$ and to the j-tuple $(S_1, \ldots, S_j)$ as its *range*. The *length* of $\mathbf{S}$ is defined to be j. If at least one of the S_i's is nonempty, define $d(\mathbf{S})$ to be the sum of the genera of the nonempty S_i; otherwise, define $d(\mathbf{S}) = 0$. A *framed* marked surface consists of a pair $(\mathbf{S}; \mathbf{k})$ where $\mathbf{S}$ is a marked surface length j and $\mathbf{k} = (k_1, \ldots, k_j)$ is a j-tuple of integers. We refer to $\mathbf{k}$ as the *framing* of $\mathbf{S}$. The *domain* of $(\mathbf{S}; \mathbf{k})$ is the pair consisting of the domain of $\mathbf{S}$ together with the j-tuple of integers $\mathbf{k}$.

A *cobordism* $\mathbf{M}$ is a compact oriented 3-manifold M along with two marked surfaces $\partial_{\text{in}}\mathbf{M}$ and $\partial_{\text{out}}\mathbf{M}$ so that the ranges of these marked surfaces form a partition of the boundary of M and the boundary orientation inherited from M on $\partial_{\text{in}}\mathbf{M}$ (respectively $\partial_{\text{out}}\mathbf{M}$) agrees (respectively disagrees) with the orientation induced by the markings.

A *framed cobordism* is a cobordism where the marked surfaces $\partial_{\text{in}}\mathbf{M}$ and $\partial_{\text{out}}\mathbf{M}$ have framings $\mathbf{k} = (k_1, \ldots, k_j)$ and $\mathbf{k}' = (k'_1, \ldots, k'_{j'})$ respectively such that $\sum_{i=1}^{j} k_i = \sum_{i=1}^{j'} k'_i$. The *degree* of a cobordism is defined to be the difference $d(\partial_{\text{out}}\mathbf{M}) - d(\partial_{\text{in}}\mathbf{M})$.

Suppose $\mathbf{M}$ and $\mathbf{N}$ are two cobordisms with

$$\partial_{\text{in}}\mathbf{M} = (S_1, \ldots, S_k; \phi_1, \ldots, \phi_k), \qquad \partial_{\text{in}}\mathbf{N} = (S'_1, \ldots, S'_k; \phi'_1, \ldots, \phi'_k) \quad \text{and}$$

$$\partial_{\text{out}}\mathbf{M} = (T_1, \ldots, T_r; \psi_1, \ldots, \psi_r), \qquad \partial_{\text{out}}\mathbf{N} = (T'_1, \ldots, T'_r; \psi'_1, \ldots, \psi'_r).$$

The cobordisms $\mathbf{M}$ and $\mathbf{N}$ are said to be *equivalent* if there is an orientation preserving homeomorphism $h\colon M \to N$ of the underlying 3-manifolds such that $h\phi_j$ is homotopic to ϕ'_j and $h\psi_j$ is homotopic to ψ'_j. Two framed cobordisms are equivalent if the underlying cobordisms are equivalent and their framings are identical.

If $\mathbf{S}$ is a marked surface of length j and $\mathbf{S}'$ is a marked surface of length j' their *ordered union*, denoted by $\mathbf{S} \sqcup \mathbf{S}'$, is the marked surface of length $j + j'$ obtained by concatenating the data for $\mathbf{S}$ (appearing first) to the data for $\mathbf{S}'$. The ordered union of framed marked surfaces is defined in a similar fashion. If $\mathbf{M}$ and $\mathbf{N}$ are two cobordisms (or framed cobordisms) then their ordered union, $\mathbf{M} \sqcup \mathbf{N}$ has the disjoint union $M \sqcup N$ as its underlying manifold; furthermore, $\partial_{\text{in}}(\mathbf{M} \sqcup \mathbf{N}) = \partial_{\text{in}}\mathbf{M} \sqcup \partial_{\text{in}}\mathbf{N}$ and $\partial_{\text{out}}(\mathbf{M} \sqcup \mathbf{N}) = \partial_{\text{out}}\mathbf{M} \sqcup \partial_{\text{out}}\mathbf{N}$.

Suppose that $\mathbf{M}$ and $\mathbf{N}$ are framed cobordisms such that the domain of $\partial_{\text{out}}\mathbf{M}$ and $\partial_{\text{in}}\mathbf{N}$ agree. We form the *composition* $\mathbf{NM}$ by using the markings associated to $\partial_{\text{out}}\mathbf{M}$ and $\partial_{\text{in}}\mathbf{N}$ in order to identify the surface underlying $\partial_{\text{out}}\mathbf{M}$ with the surface underlying $\partial_{\text{in}}\mathbf{N}$; the "in" and "out" boundaries of the composite are given by $\partial_{\text{in}}(\mathbf{NM}) = \partial_{\text{in}}\mathbf{M}$ and $\partial_{\text{out}}(\mathbf{NM}) = \partial_{\text{out}}\mathbf{N}$.

A *product* framed cobordism has as its underlying manifold the disjoint union of $\Sigma_{g(i)} \times [0, 1]$ for some j-tuple $(\Sigma_{g(1)}, \ldots, \Sigma_{g(j)})$ and markings $\phi_{i,k}\colon \Sigma_{g(i)} \times \{i\} \to \Sigma_{g(i)} \times \{i\}$ for $k = 0, 1$ and $i = 1, \ldots, j$; furthermore, the framings on the two ends agree.

Let $\mathscr{FC}$ be the category whose objects are pairs of j-tuples $((\Sigma_{g(1)}, \ldots, \Sigma_{g(j)}); \mathbf{k})$, $j \geq 0$, where $(\Sigma_{g(1)}, \ldots, \Sigma_{g(j)})$ a j-tuple of model surfaces and $\mathbf{k}$ is a j-tuple of integers. A morphism in $\mathscr{FC}$ is a framed cobordism $\mathbf{M}$ where the domain of $\mathbf{M}$ (as a morphism) is the domain of $\partial_{\text{in}}\mathbf{M}$ and the range of $\mathbf{M}$ is domain of $\partial_{\text{out}}\mathbf{M}$. This category admits the operation of ordered union previously described.

A connected cobordism is called *monotone increasing* if its "in" surface is connected, it is called *monotone decreasing* if its "out" surface is connected. An arbitrary cobordism is monotone increasing if all its connected components are monotone increasing, it is monotone decreasing if all its connected components are monotone decreasing. Let $\mathscr{FC}^+$ be the subcategory of $\mathscr{FC}$ with the same objects as $\mathscr{FC}$ and whose morphisms are monotone increasing framed cobordisms. Let $\mathscr{FC}^-$ be the subcategory of $\mathscr{FC}$ with the same objects as $\mathscr{FC}$ and whose morphisms are monotone decreasing framed cobordisms. A subcategory $\mathscr{FC}'$ of $\mathscr{FC}$ is *monotone* if it is a subcategory of $\mathscr{FC}^+$ or of $\mathscr{FC}^-$.

Let R be a commutative ring with unity and let $\mathscr{A}$ be an abelian subcategory of the category of all R-modules. Suppose $\mathscr{A}$ admits a *trace*, i.e. for each object M of $\mathscr{A}$ there is assigned an R-linear function $\text{trace}_M\colon \text{End}\,(M) \to R$ such that:

(1) If $f \in \text{Hom}(M, M')$ and $g \in \text{Hom}(M', M)$ then $\text{trace}_M(gf) = \text{trace}_{M'}(fg)$.
(2) If $f \in \text{End}(M)$ and $g \in \text{End}(M')$ then

$$\text{trace}_{M \oplus M'}(f \oplus g) = \text{trace}_M(f) + \text{trace}_{M'}(g).$$

Let $gr(\mathscr{A})$ be the category of (non-negatively) graded objects in $\mathscr{A}$ and graded morphisms of possibly nonzero degree. We associate to $gr(\mathscr{A})$ the *projectivized* category $Pgr(\mathscr{A})$; it has the same objects as $gr(\mathscr{A})$ and a morphism $M \to M'$ is an equivalence class of morphisms

$M \to M'$ of $gr(\mathscr{A})$ where two linear maps f, g: $M \to M'$ are declared to be equivalent if there is a unit (invertible element) $\lambda \in R$ such that $f = \lambda g$.

Let G be a given integer. Consider a subcategory $\mathscr{F}\mathscr{C}'$ of $\mathscr{F}\mathscr{C}$ with the following properties:

(1) For any object $((\Sigma_{g(1)}, \ldots, \Sigma_{g(j)}); \mathbf{k})$ of $\mathscr{F}\mathscr{C}'$, $g(i) > G$, $i = 1, \ldots, j$.

(2) For any $g > G$ and integer k, $\mathbf{S}_{g,k} \equiv (\Sigma_g; (k))$ is an object of $\mathscr{F}\mathscr{C}'$.

(3) For any $g, g' > G$ and integers k, k', $\mathrm{Hom}_{\mathscr{F}\mathscr{C}'}(\mathbf{S}_{g,k}, \mathbf{S}_{g',k'})$ contains every connected framed cobordism from $\mathbf{S}_{g,k}$ to $\mathbf{S}_{g',k'}$.

Definition 1.1. A *projective graded cobordism functor* is a covariant functor Q: $\mathscr{F}\mathscr{C}' \to Pgr(\mathscr{A})$ satisfying:

(1) Q takes equivalent framed cobordisms to the same morphism.
(2) There exists an integer s (fixed for a given Q) so that if $\mathbf{M}$ is a framed cobordism of degree d then $Q(\mathbf{M})$ is a graded morphism of degree ds.

In the examples constructed in §3 we also assume that $\mathscr{F}\mathscr{C}'$ is monotone; however, Theorem 1.3 below does not require this condition.

Remark. Suppose the subcategory $\mathscr{F}\mathscr{C}'$ is closed under ordered union and that the category $\mathscr{A}$ admits a tensor product. We say that the projective graded cobordism functor Q: $\mathscr{F}\mathscr{C}' \to Pgr(\mathscr{A})$ is *multiplicative* if it sends ordered unions to tensor products. Indeed, the functors Q_n which we define in §3 have this property (provided one ignores "$\mathbb{Z}$-structure").

Let M be a connected closed oriented 3-manifold and let $\zeta \in H_2(M)$ be nonzero and primitive. Given a projective graded cobordism functor, we will associate a power series to the pair (M, ζ). Let F be a connected oriented embedded surface in M representing ζ. Form a framed cobordism, $\mathbf{M}(\zeta)$, as follows. Choose a homeomorphism ϕ: $\Sigma_g \to F$ and an integer k. The manifold underlying $\mathbf{M}(\zeta)$ will be the result of cutting M along F; the homeomorphism ϕ serves as the marking of both $\partial_{\mathrm{in}}\mathbf{M}(\zeta)$ and $\partial_{\mathrm{out}}\mathbf{M}(\zeta)$; the integer k provides a framing for both ends. Let Q: $\mathscr{F}\mathscr{C}' \to Pgr(\mathscr{A})$ be a projective graded cobordism functor and suppose $\mathbf{M}(\zeta)$ as above is an object of $\mathscr{F}\mathscr{C}'$. Observe that $Q(\mathbf{M}(\zeta))$ has degree zero.

Definition 1.2. The formal power series associated to (Q, k, M, ζ) is:

$$\lambda_{Q,k}(M, \zeta)(t) = \sum_{i=0}^{\infty} (-1)^i \operatorname{trace}(Q(\mathbf{M}(\zeta))_i)\, t^i$$

where trace appearing above is the trace associated to the category $\mathscr{A}$ and where the trace of the projective morphism $Q(\mathbf{M}(\zeta))$ is calculated by choosing a representative in $gr(\mathscr{A})$.

Notice that $\lambda_{Q,k}(M, \zeta)(t)$ is defined up to a unit in R. Let $R_S[[t]]$ be the ring obtained by localizing the formal power series in t over R with respect to the multiplicative subset $S = \{t^{sk} \mid k \geq 0\}$ where s is the integer associated to Q by (2) of Definition 1.1.

THEOREM 1.3. *As an element of* $R_S[[t]]$, *the formal power series* $\lambda_{Q,k}(M, \zeta)(t)$ *is uniquely determined by* (Q, k, M, ζ) *up to multiplication by a unit of the form* ut^{sj} *where* u *is a unit of* R *and* j *is an integer.*

Proof. The following proof is a simplification suggested by Rick Litherland of the argument in [6]. Suppose μ: $\Sigma_g \to F$ is another marking of F. Let $\mathbf{M}'(\zeta)$ be the framed cobordism constructed in the same fashion as $\mathbf{M}(\zeta)$ above but using μ instead of ϕ. Let $\mathbf{A}$ and $\mathbf{B}$ be product cobordisms with underlying manifold $F \times I$. In each case the framings are given by the integer k. Let $\partial_{\mathrm{in}}\mathbf{A}$ have $F \times \{0\}$ as underlying manifold and marking μ:

$\Sigma_g \to F \times \{0\}$ and let $\partial_{\text{out}}\mathbf{A}$ have $F \times \{1\}$ as underlying manifold and marking ϕ: $\Sigma_g \to F \times \{1\}$. Let $\partial_{\text{in}}\mathbf{B}$ have $F \times \{0\}$ as underlying manifold and marking ϕ: $\Sigma_g \to F \times \{0\}$ and let $\partial_{\text{out}}\mathbf{B}$ have $F \times \{1\}$ as underlying manifold and marking μ: $\Sigma_g \to F \times \{1\}$. Observe that $\mathbf{A}\ \mathbf{M}(\zeta)\ \mathbf{B}$ is equivalent to $\mathbf{M}'(\zeta)$ and so $Q(\mathbf{M}'(\zeta)) = Q(\mathbf{A}\ \mathbf{M}(\zeta)\ \mathbf{B})$. Since the trace is invariant under cyclic permutation we have

$$\sum_{i=0}^{\infty} (-1)^i \operatorname{trace}(Q(\mathbf{M}'(\zeta))_i)\, t^i = \sum_{i=0}^{\infty} (-1)^i \operatorname{trace}(Q(\mathbf{A}\ \mathbf{M}(\zeta)\ \mathbf{B})_i)\, t^i$$

$$= \sum_{i=0}^{\infty} (-1)^i \operatorname{trace}(Q(\mathbf{M}(\zeta)\ \mathbf{B}\ \mathbf{A})_i)\, t^i.$$

Since $\mathbf{M}(\zeta)\mathbf{BA}$ is equivalent to $\mathbf{M}(\zeta)$, the last power series above is equal to $\sum_{i=0}^{\infty}(-1)^i \operatorname{trace}(Q(\mathbf{M}(\zeta))_i)\, t^i$ and so $\lambda_{Q,k}(M, \zeta)(t)$ is independent of the choice of marking. Next, we show it is independent of the choice of F representing ζ. Suppose F_a and F_b are two connected oriented surfaces representing ζ. Then there exists a sequence of connected oriented surfaces $F_0, F_1, \ldots, F_n$ with $F_0 = F_a$ and $F_n = F_b$ with all F_i representing ζ and genus $(F_i) \geq \min\{\text{genus}(F_a), \text{genus}(F_b)\}$ such that $F_i \cap F_{i+1} = \emptyset$ for $i = 0, \ldots, n-1$. This can be seen by means of Cerf theoretic arguments or, more simply, by using double curve surgery to alter the surfaces F_a and F_b. Suppose F and F' both represent ζ and are disjoint. Then F' lies inside the result, M', of cutting M open along F. Cutting along F' decomposes M' into two pieces, N_1 and N_2. Choose markings for F and F' and use the integer k as a framing for all boundary components. This procedure yields framed cobordisms $\mathbf{M}$ (M cut along F) and $\mathbf{N}_1$ and $\mathbf{N}_2$ with $\mathbf{N}_1\mathbf{N}_2$ equivalent to $\mathbf{M}$. Then

$$\operatorname{trace}(Q(\mathbf{M})_i) = \operatorname{trace}(Q(\mathbf{N}_1\mathbf{N}_2)_i) = \operatorname{trace}(Q(\mathbf{N}_2\mathbf{N}_1)_{i+s(\text{genus}(F)-\text{genus}(F'))})$$

is equal to the trace cobordism obtained by cutting M along F'. Proceeding through the sequence of F_i's as above shows that $\lambda_{Q,k}(M, \zeta)(t)$ is independent of the choice of F up to multiplication by a unit of the form ut^{sj}, $u \in R$. □

§2. INTERSECTIONS IN REPRESENTATION VARIETIES

In this section we introduce notions from the topology of stratified spaces in an appropriate context. Our development of intersection homology theory involves the use of perversities defined on product spaces that depend on the stratum instead of just the codimension of the stratum. Indeed, the need to employ stratum dependent perversities already arsies in Goresky and MacPherson's treatment, [9], of the Lefschetz fixed point theorem for intersection homology; our requirement for these more general perversities arises for similar reasons.

After defining "special perversity", we describe a general setting for slant products on stratified spaces. Next, we describe the stratifications of the spaces that interest us. Finally, we compute dimensions of strata and show that a certain cycle has the correct perversity for our purposes (see Proposition 2.3).

Let X be a compact polyhedron and let Λ be a finite partially ordered set with a unique maximal element $\lambda_0 \in \Lambda$. A *filtration* of X indexed by Λ consists of a family $\{X_\lambda \mid \lambda \in \Lambda\}$ of closed subpolyhedra of X such that:

(1) $X = X_{\lambda_0}$
(2) $X_\lambda \cap X_\mu = \bigcup_{\nu \leq \lambda; \nu \leq \mu} X_\nu$

A consequence of the second condition is that $X_\lambda \subset X_\mu$ whenever $\lambda \leq \mu$. We refer to X as a *filtered polyhedron*. If X is a filtered polyhedron with indexing set Λ and Y is another filtered polyhedron with indexing set Λ' then a morphism $X \to Y$ consists of a PL map $f: X \to Y$ and an order preserving set map $\phi: \Lambda \to \Lambda'$ such that $f(X_\lambda) \subset Y_{\phi(\lambda)}$ for all $\lambda \in \Lambda$. Define $\mathring{X}_\lambda \equiv X_\lambda - \bigcup_{\mu<\lambda} X_\mu$. A filtration is said to be *regular* if for each $\lambda \in \Lambda$ such that $\mathring{X}_\lambda$ is nonempty:

(1) closure $(\mathring{X}_\lambda) = X_\lambda$
(2) $\mathring{X}_\lambda$ is a manifold.

Note that the connected components of $\mathring{X}_\lambda$ have the same dimension. The sets $\mathring{X}_\lambda$ are called the *strata* of the regular filtration. The *codimension* of a nonempty stratum X_λ, denoted $k(\lambda)$, is defined to be $k(\lambda) = \dim X - \dim X_\lambda$. A *stratification* of X is a filtration of X that is locally trivial in the following sense. Given $x \in \mathring{X}_\lambda$ there exists a compact polyhedron L with a filtration $\{L_\mu\}$ indexed by $\Lambda(\lambda) = \{\mu \mid \lambda < \mu\}$ and a PL homeomorphism of filtered polyhedra $D^m \times \mathrm{cone}(L)$ to a neighborhood of $x \in X$. Here D^m is an m-disk where $m = \dim \mathring{X}_\lambda$ and $\mathrm{cone}(L)$ is the cone on L. In particular, a stratification is a regular filtration. A *stratified space* is a space equipped with a stratification. Recall that a *pseudomanifold of dimension n* is a compact polyhedron X for which there exists a closed subspace Σ with $\dim \Sigma \leq n-2$ such that $X - \Sigma$ is an n-dimensional manifold which is dense in X; X is *oriented* if $X - \Sigma$ is oriented. A stratified space with the property that each nonmaximal stratum has codimension at least two is a pseudomanifold.

Let X be a stratified space and let P, Q be subpolyhedra. We say that P and Q are *dimensionally transverse* if for all λ, $P \cap \mathring{X}_\lambda$ and $Q \cap \mathring{X}_\lambda$ are in general position in the manifold $\mathring{X}_\lambda$. By a theorem of McCrory (see [19]) any P, Q can be made dimensionally transverse by a small perturbation of P so that the dimension of $P \cap X_\lambda$ is preserved for all λ; furthermore, if $C \subset P$ is a closed subset along which P and Q are dimensionally transverse then the perturbation can be chosen to be the identity along C.

Let Λ be a finite partially ordered set with a unique maximal element λ_0. A *loose perversity* is a function $p: \Lambda \to \{0, 1, 2, \ldots\}$ (compare [13, §2]). Let X be a stratified space indexed by Λ. A PL map from an i-simplex $\sigma: \Delta^i \to X$ is *p-allowable* if for all λ, $\sigma^{-1}(\mathring{X}_\lambda)$ is contained in the $i - k(\lambda) + p(\lambda)$ skeleton of Δ^i. A singular PL chain is p-allowable if each its singular simplices is p-allowable. Let $IS^p_*(X)$ be the subcomplex of the chain complex of all singular PL chains consisting of those chains τ such that both τ and $\partial\tau$ are p-allowable. Define intersection homology groups by $IH^p_*(X) = H_*(IS^p_*(X))$. For an arbitrary loose perversity these groups can depend on the stratification of X. When X is compact, standard techniques show that the groups $IH^p_*(X)$ defined above are isomorphic to those defined in [8], (see [13]).

Remark. Among other restrictions, a perversity in the sense of [8] is a function of the codimension of the stratum rather than of the stratum. We want to use perversities that may take on different values on strata having the same codimension.

We say that the pair (X, p) consisting of a stratified space X and loose perversity p is *permissible* if $p(\lambda_0) = 0$ and $p(\lambda) - k(\lambda) < -1$ for all $\lambda < \lambda_0$. Observe that if (X, p) is a permissible pair then X cannot have nonmaximal strata with codimension less than two; furthermore, any $\eta \in IH^p_i(X)$ can be represented by a PL embedded oriented i-pseudomanifold. Suppose that X is an oriented m-pseudomanifold and p, q are loose perversities such that (X, p) and (X, q) are permissible. If $p + q$ is such that $(X, p + q)$ is permissible then we can define a pairing $IH^p_i(X) \otimes IH^q_j(X) \to IH^{p+q}_{i+j-m}(X)$ as follows: represent the two classes to be paired by oriented pseudomanifolds, make them dimensionally transverse, and then take their oriented intersection. Since $(X, p+q)$ is permissible, all parts of the intersection coming from singular strata have at least codimension two in the intersection thus assuring that the intersection is an oriented pseudomanifold. It is easily seen that, with the appropriate sign conventions, this pairing coincides with the one defined in [8], provided we restrict our attention to the perversities used there.

The spaces whose intersection homology is of most concern to us will have only strata of even codimension. In this case there are three perversities that we will use. Let Λ be a partially ordered indexing set with unique maximal element λ_0. Given a stratified space X with codimension function $k\colon \Lambda \to \{0, 2, 4, \ldots\}$, define the *middle perversity*, $m\colon \Lambda \to \{0, 1, 2, \ldots\}$, by:

$$m(\lambda) = \begin{cases} 0 & \text{if } \lambda = \lambda_0 \\ \frac{k(\lambda)}{2} - 1 & \text{if } \lambda < \lambda_0. \end{cases}$$

The next two perversities are defined on a product of two stratified spaces with the product stratification. Let M be a partially ordered indexing set with unique maximal element μ_0 and Y a stratified space indexed by M with codimension function $k'\colon M \to \{0, 2, 4, \ldots\}$. Partially order $\Lambda \times M$ lexicographically and give $X \times Y$ the product stratification indexed by $\Lambda \times M$. The *special perversity*, $s\colon \Lambda \times M \to \{0, 1, 2, \ldots\}$, is given by:

$$s(\lambda, \mu) = \begin{cases} 0 & \text{if } (\lambda, \mu) = (\lambda_0, \mu_0) \\ \frac{k(\lambda)}{2} - 1 & \text{if } \mu = \mu_0 \text{ and } \lambda < \lambda_0 \\ \frac{k'(\mu)}{2} - 1 & \text{if } \lambda = \lambda_0 \text{ and } \mu < \mu_0 \\ \frac{k(\lambda) + k'(\mu)}{2} - 1 & \text{otherwise.} \end{cases}$$

Finally, we define $r\colon \Lambda \times M \to \{0, 1, 2, \ldots\}$ by:

$$r(\lambda, \mu) = \begin{cases} 0 & \text{if } \lambda = \lambda_0 \\ \frac{k(\lambda)}{2} - 1 & \text{if } \lambda < \lambda_0. \end{cases}$$

Observe that $(X \times Y, r+s)$ is permissible. Hence there is a well defined pairing:

$$IH^r_{i+\dim Y}(X \times Y) \otimes IH^s_j(X \times Y) \to IH^{r+s}_{i+j-\dim X}(X \times Y).$$

The natural projection $p\colon X \times Y \to Y$ is a PL mapping. Suppose $Z \subset X \times Y$ is an embedded pseudomanifold such that Z is $(r+s)$-allowable. Then, after a small perturbation, $p(Z)$ will be an m-allowable embedded pseudomanifold. Consequently:

Theorem 2.1. *Each $\zeta \in IH^s_j(X \times Y)$ induces a homomorphism $\zeta_*\colon IH^m_i(X) \to IH^m_{i+j-\dim X}(Y)$; furthermore, if $\eta \in IH^s_k(Y \times Z)$ then $\eta_* \zeta_*$ is induced by the homology class $\eta \cdot \zeta$ determined as follows: Let A and B be pseudomanifolds representing ζ and η respectively. Consider $A \times Z$ and $X \times B$ in $X \times Y \times Z$. Make them dimensionally transverse, take their intersection, and project to $X \times Z$.*

Proof. Represent $\kappa \in IH^m_i(X)$ by an embedded oriented pseudomanifold K. Make $K \times Y$ dimensionally transverse to an oriented pseudomanifold C representing ζ. Form the oriented intersection $(K \times Y) \cap C$ and project to Y. This defines ζ_*. The second statement is a standard argument in theory of correspondences (see [7]); it is easily verified that the perversities are correct. $\square$

Recall that the projective unitary group, $PU(n)$ is defined to be $U(n)$ modulo its center. Let F be a closed orientable surface of genus $g > 0$. Choose a basepoint for F. Let $S_{n,k}(F)$ be the space of representations $\rho\colon \pi_1(F) \to PU(n)$ so that any associated $\mathbb{C}^n$-bundle over F has first Chern number congruent to k modulo n. Let $P_{n,k}(F)$ be the quotient $S_{n,k}(F)$ under the conjugation action of $PU(n)$. If F is given a complex structure then $P_{n,k}(F)$ can be canonically identified with the coarse moduli space of semistable holomorphic $\mathbb{P}^{n-1}$-bundles of degree k modulo n over F. In particular, it is a stratified space all of whose strata are even dimensional.

Let a_i, b_i be a standard generating set for $\pi_1(F)$ chosen so that the intersection number $a_i \cdot b_i = +1$. Let $\Gamma(F)$ be the central extension of the fundamental group of F with presentation

$$\Gamma(F) = \langle a_i, b_i, z \mid \prod_{i=1}^{g} [a_i, b_i] = z, z \text{ is central} \rangle$$

where $[a_i, b_i]$ is the commutator a and b. Alternately, $\Gamma(F)$ is the fundamental group of a principal $U(1)$-bundle over F having first Chern number 1. Let

$$R_{n,k}(F) = \{\rho \colon \Gamma(F) \to SU(n) \mid \rho(z) = \omega^k\}$$

where ω is the element of $SU(n)$ given by $e^{2\pi i/n}$ times the identity matrix. Note that ω generates the center of $SU(n)$. Let $X_{n,k}(F)$ be the quotient of $R_{n,k}(F)$ by the action of $SU(n)$ by conjugation. The universal covering homomorphism $p \colon SU(n) \to PU(n)$ induces covering maps $p \colon R_{n,k}(F) \to S_{n,k}(F)$ and $p \colon X_{n,k}(F) \to P_{n,k}(F)$. The group of deck transformations in either case is $H^1(F, \mathbb{Z}/n)$.

A *hollow handlebody* is a compact irreducible orientable 3-manifold H with a preferred boundary component F such that $\pi_1(F) \to \pi_1(H)$ is a surjection. A hollow handlebody is a handlebody if it has only one boundary component; in this case its genus is defined to be the genus of its boundary.

Let H be a hollow handlebody with preferred boundary surface F and other boundary surfaces F_i, $i = 1, \ldots, r$ where the genus of F is g and the genus of F_i is g_i. Up to homeomorphism, H is the result of adding a system of 2-handles $N(D_j^2)$ to $F \times [0, 1]$ along $F \times \{1\}$. For each 2-handle $N(D_j^2) \cong D_j^2 \times [0, 1]$ choose an arc $\{v_j\} \times [0, 1]$ where v_j lies in the interior of D_j^2. Then $H' = \bigcup_i F_i \bigcup \bigcup_j \{v_j\} \times [0, 1]$ is a strong deformation retract of H. Choose a basepoint for H', a basepoint for F, basepoints for each F_i, and basepaths connecting the basepoint of H' to the other basepoints. Applying Van Kampen's theorem, we have an isomorphism $\pi_1(H) \cong \pi_1(H') \cong (\bigstar_{i=1}^r \pi_1(F_i)) \star F(d)$ where $F(d)$ is the free group of rank $d = g - \sum_{i=1}^r g_i$. For a sequence of integers $\mathbf{k} = (k_1, \ldots, k_r)$ define $S_{n,\mathbf{k}}(H)$ to be those $PU(n)$ representations of $\pi_1(H)$ whose restriction to the subgroup $\pi_1(F_i)$ lies in $S_{n,k_i}(F_i)$. If $k = \sum_{i=1}^r k_i$, then restriction yields a map

$$S_{n,\mathbf{k}}(H) \to S_{n,k}(F) \times \left(\prod_{i=1}^{r} S_{n,k_i}(F_i) \right).$$

This map depends on the choice of basepoints and basepaths; however, taking the quotient by the conjugation action yields a map

$$P_{n,\mathbf{k}}(H) \xrightarrow{\iota} P_{n,k}(F) \times \left(\prod_{i=1}^{r} P_{n,k_i}(F_i) \right).$$

which is independent of these choices. Since $\pi_1(F) \to \pi_1(H)$ is surjective, the map ι is an embedding which we view as an inclusion of a subspace. Let $X_{n,\mathbf{k}}(H)$ be the inverse image of $P_{n,\mathbf{k}}(H)$ under the covering map:

$$p \colon X_{n,k}(F) \times \left(\prod_{i=1}^{r} X_{n,k_i}(F_i) \right) \to P_{n,k}(F) \times \left(\prod_{i=1}^{r} P_{n,k_i}(F_i) \right).$$

Notice that since $\pi_1(H)$ is a free product we have that $S_{n,\mathbf{k}}(H) \cong (\prod_{i=1}^r S_{n,k_i}(F_i)) \times PU(n)^d$. Provided that the genus of F and of the F_i's are all strictly greater than one, it follows that $P_{n,\mathbf{k}}(H)$ is a connected, orientable pseudomanifold. Note that $X_{n,\mathbf{k}}(H)$ has several connected components each one of which is a covering space of $P_{n,\mathbf{k}}(H)$.

In the subsequent discussion H will be a hollow handlebody as above with preferred boundary surface F and other boundary surfaces F_i, $i = 1, \ldots, r$ where the genus of F is g and the genus of F_i is g_i. Let $d = g - \sum_{i=1}^r g_i$. Let $\mathbf{k} = (k_1, \ldots, k_r)$ be an r-tuple of integers. Let $k = \sum_{i=1}^r k_i$. We will assume the following:

Hypothesis. There is at least one non-preferred surface F_i. The genus, g, of F and the genus, g_i, of F_i for $i = 1, \ldots, r$ (where $r \geq 1$) are all strictly greater than one.
We will show that the orientable pseudomanifold

$$P_{n,\mathbf{k}}(H) \subset P_{n,k}(F) \times \left(\prod_{i=1}^r P_{n,k_i}(F_i)\right)$$

is s-allowable.

We now describe stratifications of $X_{n,k}(F)$ and $\prod_{i=1}^r X_{n,k_i}(F_i)$. These spaces are finite covering spaces of $P_{n,k}(F)$ and $\prod_{i=1}^r P_{n,k_i}(F_i)$ respectively; furthermore, it will be clear that the stratifications of $X_{n,k}(F)$ and $\prod_{i=1}^r X_{n,k_i}(F_i)$ which we consider descend to stratifications of $P_{n,k}(F)$ and $\prod_{i=1}^r P_{n,k_i}(F_i)$ under the natural covering projections.

Let Γ be a group and let $E\colon \Gamma \to SU(n)$ be a representation which decomposes as a direct sum of representations $E \cong m_1 E_1 \oplus \cdots \oplus m_l E_l$, where m_i denotes the multiplicity of the irreducible factor E_i and for $i \neq j$ the representations E_i and E_j are inequivalent. Let n_i be the rank of the summand E_i. In addition, assume that the decomposition is written so that $n_1 \leq n_2 \leq \cdots \leq n_l$ and if $n_j = n_{j+1}$ then $m_j \leq m_{j+1}$. The sequence of pairs of integers $\{(n_i, m_i)\}$ will be called the *type* of the representation E.

We now specialize to the case where Γ is a centrally extended surface group as previously described. The open stratum $F[\{(n_i, m_i)\}]$ of $X_{n,k}(F)$ is defined to be the set all equivalence classes of representations of type $\{(n_i, m_i)\}$. For example, $X_{2,0}(F)$ has three strata: $F[\{(2,1)\}]$, the irreducible representations; $F[\{(1,1),(1,1)\}]$, the abelian representations; and $F[\{(1,2)\}]$, the central representations.

There are some restrictions on the sequence of pairs $\{(n_i, m_i)\}$ if the corresponding stratum is to be nonempty. We require that n_i and m_i be positive integers and that $\sum_i n_i m_i = n$. There is also the following number theoretic condition having to do with Chern classes. If $b \in \mathbb{Z}/n$ define $\operatorname{ord}(b)$ to the order of the subgroup of $\mathbb{Z}/n$ generated by b. The stratum $F[\{(n_i, m_i)\}]$ is nonempty if and only if $\operatorname{ord}(k)$ divides each n_i. For example, the space $X_{4,2}(F)$ has nonempty strata $F[\{(4,1)\}]$, $F[\{(2,1),(2,1)\}]$ and $F[\{(2,2)\}]$. In this instance, the stratum $F[\{(1,2),(2,1)\}]$ is empty.

Note that the dimension of $F[\{(n_i, m_i)\}]$ is $(\sum_i n_i^2(2h-2)+2) - 2h$ where h is the genus of F. The top stratum is $F[\{(n,1)\}]$ and its dimension is $(n^2-1)(2h-2)$.

To stratify $\prod_{i=1}^r X_{n,k_i}(F_i)$ (respectively $\prod_{i=1}^r P_{n,k_i}(F_i)$) we take as strata the product of strata of the various factors. Such strata are labelled $\mathbf{F}[\{(n_j(i), m_j(i))\}]$, where, for $i = 1, \ldots, r$, the i-th sequence indexes a stratum in the factor $X_{n,k_i}(F_i)$ (respectively $P_{n,k_i}(F_i)$). We will use the shorthand notation $\{(p_\alpha, q_\alpha), \{(n_j(i), m_j(i))\}\}$ for the open stratum $F[\{(p_\alpha, q_\alpha)\}] \times \mathbf{F}[\{(n_j(i), m_j(i))\}]$ of the product $X_{n,k}(F) \times (\prod_{i=1}^r X_{n,k_i}(F_i))$ (respectively $P_{n,k}(F) \times \prod_{i=1}^r P_{n,k_i}(F_i))$). The space $X_{n,\mathbf{k}}(H)$ (respectively $P_{n,\mathbf{k}}(H)$) acquires a regular filtration via intersection with the stratification of $X_{n,k}(F) \times (\prod_{i=1}^r X_{n,k_i}(F_i))$ (respectively $P_{n,k}(F) \times \prod_{i=1}^r P_{n,k_i}(F_i))$).

LEMMA 2.2. *The part of* $P_{n,k}(H)$ *lying in the open stratum* $\{(n,1),\{(n_j(i),m_j(i))\}\}$ *of* $P_{n,k}(F)\times(\prod_{i=1}^r P_{n,k_i}(F_i))$ *has dimension*

$$\sum_{i=1}^r \sum_j (n_j(i)^2(2g_i-2)+2) - \sum_{i=1}^r 2g_i + rn^2 - \sum_{i=1}^r \sum_j m_j(i)^2 + d(n^2-1) - (n^2-1).$$

Proof. Recall that $\pi_1(H)$ is isomorphic to the free product $(\bigstar_{i=1}^r \pi_1(F_i)) \star F(d)$. Hence the space of representations of $\pi_1(H)$ into $PU(n)$ satisfying the restrictions in type given by the open stratum will be the product of the spaces of representations of the $\pi_1(F_i)$ and $F(d)$ satisfying those restrictions.

The stratum $F_i[\{(n_j(i),m_j(i))\}]$ of $P_{n,k_i}(F_i)$ has dimension $\sum_j (n_j(i)^2(2g_i-2)+2)-2g_i$. The space of representations giving rise to this stratum has dimension

$$\sum_j (n_j(i)^2(2g_i-2)+2) - 2g_i + n^2 - \sum_j m_j(i)^2.$$

The term $n^2-\sum_j m_j(i)^2$ in the above expression is the dimension of $PU(n)/\mathrm{Stab}$ where Stab is the stabilizer of a representation of type $\{(n_j(i),m_j(i))\}$. Since we place no restrictions on where the generators of the free group go, it follows that

$$\sum_{i=1}^r \sum_j (n_j(i)^2(2g_i-2)+2) - \sum_{i=1}^r 2g_i + rn^2 - \sum_{i=1}^r \sum_j m_j(i)^2.$$

is the dimension of the space of representations of $\pi_1(H)$ into $PU(n)$ that lie over the space of conjugacy classes we are interested in. Finally, we are only concerned with those representations that are irreducible hence the decrease in dimension due to passing to the quotient by the conjugation action is n^2-1. This yields the desired formula. □

Observe that if we subtract away one half of the dimension of the stratum $\{(n,1),\{(n_j(i),m_j(i))\}\}$ we obtain:

$$\sum_{i=1}^r \sum_j (n_j(i)^2(g_i-1)+1) - \sum_{i=1}^r g_i + rn^2 - \sum_{i=1}^r \sum_j m_j(i)^2 + d(n^2-1)$$
$$-(n^2-1)-(n^2(g-1)+1-g).$$

Using the fact that $\sum_j n_j(i)m_j(i)=n$ and $g=\sum_{i=1}^r g_i + d$, we obtain:

$$\sum_{i=1}^r \sum_j (n_j(i)^2(g_i-1)+1) - \sum_{i=1}^r g_i + \sum_{i=1}^r \left(\sum_j n_j(i)m_j(i)\right)^2 - \sum_{i=1}^r \sum_j m_j(i)^2 + dn^2 - d - n^2$$
$$+1-n^2\left(\sum_{i=1}^r g_i + d - 1\right) - 1 + \sum_{i=1}^r g_i + d$$
$$= \sum_{i=1}^r \sum_j (n_j(i)^2(g_i-1)+1) - \sum_{i=1}^r \left(\sum_j n_j(i)m_j(i)\right)^2 (g_i-1) - \sum_{i=1}^r \sum_j m_j(i)^2.$$

It is clear that this last number is negative unless the representations of the fundamental groups of all of the F_i's are irreducible in which case it is zero.

In general, if we look at the part of $P_{n,k}(H)$ lying in the stratum of the form $\{(p_\alpha,q_\alpha),\{(n_j(i),m_j(i))\}\}$ then, since the fundamental group of H is carried by F, we can partition the $(n_j(i),m_j(i))$'s so that the subsets add up to (p_α,q_α) for each α. Consequently, the general computation can be achieved by summing the answers over the (p_α,q_α)'s for representation spaces and then taking the quotient by the conjugation action. This yields:

PROPOSITION 2.3. *The space* $P_{n,k}(H)$ *is an s-allowable orientable pseudomanifold in* $P_{n,k}(F)\times(\prod_{i=1}^{r} P_{n,k_i}(F_i))$ *of half dimension.*

§3. CONSTRUCTION OF THE FUNCTORS

In this section we define the functors that were promised in §1. The procedure for assigning a morphism to a cobordism involves some choices. We show that the morphism that we define is independent of these choices; see Theorem 3.1. The proof that composition of cobordisms corresponds to composition of morphisms will be deferred to §4.

Let $\mathscr{FC}_{(2)}^{+}$ be the full subcategory of $\mathscr{FC}^{+}$ whose objects are pairs of tuples $((\Sigma_{g(1)},\ldots,\Sigma_{g(r)});(k_1,\ldots,k_r))$ where $r>0$ and for all i, $g(i)>1$. Define $\mathscr{FC}_{(2)}^{-}$ similarly. Let $\mathscr{FC}'$ be one of the categories $\mathscr{FC}_{(2)}^{+}$ or $\mathscr{FC}_{(2)}^{-}$.

In what follows our base ring will be the integers. Let $\mathscr{V}_{\mathbb{Z}}$ be the category of finite dimensional rational vector spaces with "$\mathbb{Z}$-structure". Its objects are triples (B', B, θ) where B is a finite dimensional rational vector space, B' is a finitely generated abelian group, and θ: $B'\otimes\mathbb{Q}\to B$ is an isomorphism. We refer to B' as the *$\mathbb{Z}$-structure* of B. A morphism $(B_1', B_1, \theta_1)\to(B_2', B_2, \theta_2)$ is a $\mathbb{Q}$-linear map $f\colon B_1\to B_2$ such that for some homomorphism $f'\colon B_1'\to B_2'$, $f\theta_1=\theta_2(f'\otimes\mathrm{id})$. This category admits a trace, namely the usual trace of an endomorphism of a finite dimensional vector space; observe that the trace of an endomorphism in $\mathscr{V}_{\mathbb{Z}}$ is an integer. Let $Pgr(\mathscr{V}_{\mathbb{Z}})$ be the associated projectivized graded category. For all positive integers n, we will define projective graded cobordism functors Q_n: $\mathscr{FC}'\to Pgr(\mathscr{V}_{\mathbb{Z}})$. The functor Q_n is defined on objects by:

$$Q_n((\Sigma_{g(1)},\ldots,\Sigma_{g(r)});(k_1,\ldots,k_r)) = IH_*^m\left(\prod_{i=1}^{r} P_{n,k_i}(\Sigma_{g(i)})\right)$$

where IH_*^m is mid-perversity intersection homology with rational coefficients. The underlying $\mathbb{Z}$-structure is given by intersection homology with integral coefficients. Next, we must assign to a morphism in $\mathscr{FC}'$ a morphism in $Pgr(\mathscr{V}_{\mathbb{Z}})$. The restriction that $\mathbf{M}$ be monotone in the sense of §1 is required in order to assure that Q_n is a functor. For the purpose of the subsequent discussion, assume that $\mathbf{M}$ is a connected framed cobordism with underlying manifold M. Nonconnected cobordisms will be dealt with by taking the tensor product of morphisms over the connected components. Let $(F_1,\ldots,F_r)$ be the surfaces underlying $\partial_{\text{in}}\mathbf{M}$ with markings $\phi_i\colon\Sigma_{g(i)}\to F_i$, $i=1,\ldots,r$, and framing $(k_1,\ldots,k_r)$. Let $(G_1,\ldots,G_s)$ be the surfaces underlying $\partial_{\text{out}}\mathbf{M}$ with markings $\psi_i\colon\Sigma_{h(i)}\to G_i$, $i=1,\ldots,s$, and framing $(l_1,\ldots,l_s)$. Recall that we assume $r,s>0$ and $g(i),h(j)>1$ for all i,j. By the definition of a framed cobordism we have that $\Sigma_{i=1}^{r} k_i=\Sigma_{j=1}^{s} l_j$. Given a Heegaard surface F for M, we define an s-allowable pseudomanifold

$$P(\mathbf{M},F)\subset\left(\prod_{i=1}^{r} P_{n,k_i}(\Sigma_{g(i)})\right)\times\left(\prod_{j=1}^{s} P_{n,l_j}(\Sigma_{h(j)})\right)$$

of half dimension. We will show that the homology class represented by $P(\mathbf{M},F)$ is independent of F. Hence there is a well defined morphism

$$P(\mathbf{M})_*\colon IH_*^m\left(\prod_{i=1}^{r} P_{n,k_i}(\Sigma_{g(i)})\right)\to IH_*^m\left(\prod_{j=1}^{s} P_{n,l_j}(\Sigma_{h(j)})\right)$$

obtained by applying Theorem 2.1. If $\mathbf{M}$ is a morphism of $\mathscr{FC}'$ we define $Q_n(\mathbf{M})=P(\mathbf{M})_*$. The fact that $P(\mathbf{M},F)$ is half dimensional implies that the dimension shift of Q_n is given by $d(\mathbf{M})(n^2-1)$; hence, the number s associated to Q_n by Definition 1.1 is n^2-1.

Let h: $M \to \mathbb{R}$ be a Morse function with only index 1 and index 2 critical points. Assume that $h^{-1}(0) = \bigcup_{i=1}^{r} F_i$, $h^{-1}(1) = \bigcup_{j=1}^{s} G_j$ and the values of all index i critical points lie in the open interval $(\frac{i-1}{2}, \frac{i}{2})$. Then $F = h^{-1}(\frac{1}{2})$ is a Heegaard surface. Observe that if $H_1 = h^{-1}([0, \frac{1}{2}])$ and $H_2 = h^{-1}([\frac{1}{2}, 1])$ then the H_i's are hollow handlebodies with preferred surface F. In fact, if $F \subset M$ is any closed surface such that there are hollow handlebodies H_1, H_2 with F as preferred surface and so that $H_1 \cup H_2 = M$, $H_1 \cap H_2 = F$, $\bigcup_{i=1}^{r} F_i \subset H_1$, $\bigcup_{j=1}^{s} G_j \subset H_2$ then F is a Heegaard surface.

Let T be a compact surface of genus one with one boundary component which is (locally flatly and properly) embedded in a ball B so that $B = H'_1 \cup H'_2$, $H'_1 \cap H'_2 = T$ and $\pi_1(T)$ surjects onto $\pi_1(H'_i)$ for $i = 1, 2$. It is an elementary exercise in 3-manifold topology to show that the pair (B, T) is unique up to homeomorphism. If $F \subset M$ is a Heegard surface and $B \subset M$ is a ball such that ∂B is transverse to F and such that $F \cap B$ is a disk then a new Heegaard surface F' can be constructed by removing $F \cap B$ and replacing it by T. This operation, called *stabilization*, is well defined up to homeomorphism of pairs (M, F). Using Cerf theory, one can see that if F_1, F_2 are two Heegaard surfaces for M then there is a third Heegard surface F_3 so that (M, F_3) is topologically equivalent to the result of stabilizing (M, F_2) several times.

Given the framed cobordism $\mathbf{M}$ as above, choose a Heegard surface F for the underlying manifold M. Choose a marking η: $\Sigma_g \to F$. Give F the framing $k = \sum_{i=1}^{r} k_i = \sum_{j=1}^{s} l_j$. Use this data to make H_1 into a framed cobordism $\mathbf{N}_1$ with $\partial_{\text{in}} \mathbf{N}_1 = \partial_{\text{in}} \mathbf{M}$ and $\partial_{\text{out}} \mathbf{N}_1 = (F; \eta; (k))$. Let $\mathbf{N}_2$ be the framed cobordism with H_2 as its underlying manifold and $\partial_{\text{in}} \mathbf{N}_2 = (F; \eta; (k))$ and $\partial_{\text{out}} \mathbf{N}_2 = \partial_{\text{out}} \mathbf{M}$. By Proposition 2.3, $P_{n,k}(H_1)$ is an s-allowable pseudomanifold in $(\prod_{i=1}^{r} P_{n,k_i}(F_i)) \times P_{n,k}(F)$ and $P_{n,k}(H_2)$ is an s-allowable pseudomanifold in $P_{n,k}(F) \times (\prod_{i=1}^{r} P_{n,l_j}(G_j))$. Use the markings to map $P_{n,k}(H_1)$ and $P_{n,k}(H_2)$ to s-allowable orientable pseudomanifolds $P(\mathbf{N}_1)$ and $P(\mathbf{N}_2)$ in $(\prod_{i=1}^{r} P_{n,k_i}(\Sigma_{g(i)}) \times P_{n,k}(\Sigma_g)$ and $P_{n,k}(\Sigma_g) \times (\prod_{j=1}^{s} P_{n,l_j}(\Sigma_{h(j)})$ respectively. By Theorem 2.1, $P(\mathbf{N}_1)$ and $P(\mathbf{N}_2)$ induce homomorphisms:

$$P(\mathbf{N}_1)_*: IH_*^m\left(\prod_{i=1}^{r} P_{n,k_i}(\Sigma_{g(i)})\right) \to IH_*^m(P_{n,k}(\Sigma_g))$$

$$P(\mathbf{N}_2)_*: IH_*^m(P_{n,k}(\Sigma_g)) \to IH_*^m\left(\prod_{j=1}^{s} P_{n,l_j}(\Sigma_{h(j)})\right).$$

Let $P(\mathbf{M})_* = P(\mathbf{N}_2)_* P(\mathbf{N}_1)_*$.

THEOREM 3.1. *Up to sign, the homomorphism $P(\mathbf{M})_*$ is independent of the choices made.*

Proof. Suppose F and F_1 are equivalent Heegaard surfaces. Then there is a homeomorphism h: $M \to M$ which restricts to the identity on ∂M and $h(F) = F_1$. The map h induces an equivalence between the associated representation theoretic objects and so $\pm P(\mathbf{M})_*$ is independent of the choice of F within its equivalence class. The choice of marking of F also has no effect on $\pm P(\mathbf{M})_*$. Let η': $\Sigma_g \to F$ be another choice of marking and let $\gamma = \eta^{-1}\eta'$. The induced map γ_*: $P_{n,k}(\Sigma_g) \to P_{n,k}(\Sigma_g)$ determines a change of coordinates $\mathrm{id} \times \gamma_* \times \mathrm{id}$: $A \times P_{n,k}(\Sigma_g) \times B \to A \times P_{n,k}(\Sigma_g) \times B$ where $A = \prod_{i=1}^{r} P_{n,k_i}(\Sigma_{g(i)})$ and $B = \prod_{j=1}^{s} P_{n,l_j}(\Sigma_{h(j)})$ so that if $P(\mathbf{N}'_1)$ and $P(\mathbf{N}'_2)$ are the pseudomanifolds associated to the choice η' then

$$\mathrm{id} \times \gamma_* \times \mathrm{id}((P(\mathbf{N}_1) \times B) \cap (A \times P(\mathbf{N}_2))) = (P(\mathbf{N}'_1) \times B) \cap (A \times P(\mathbf{N}'_2)).$$

Since this change of coordinates is invisible to projection to $A \times B$, we see that $P(\mathbf{M})$ is independent of the marking. A standard dimension count shows that the intersection class determined by $P(\mathbf{M})$ is independent of the choice of perturbation used to make $P(\mathbf{N}_1)$ and $P(\mathbf{N}_2)$ dimensionally transverse.

Next we must address the choice of equivalence class of Heegaard splitting. By the Reidemeister-Singer theorem it is sufficient to show that if F' is the result of stabilizing F once then, up to sign, $P(\mathbf{M}, F)$ and $P(\mathbf{M}, F')$ represent the same intersection homology class. Choose a pinch map $p\colon \Sigma_{g+1} \to \Sigma_g$ which is basepoint and orientation preserving. Also assume that there is a simple closed curve C on Σ_{g+1} cutting off a genus one subsurface $T \subset \Sigma_{g+1}$ so that the restriction of p to $\Sigma_{g+1} - T$ is an embedding and $p(T)$ is a point. The marking $\phi'\colon \Sigma_{g+1} \to F'$ should be chosen so that there is a disk $D \subset \Sigma_g$ and a regular neighborhood $N(T)$ so that $p(\Sigma_{g+1} - N(T)) = \Sigma_g - D$ and the restrictions of ϕp and ϕ' to $\Sigma_{g+1} - N(T)$ agree (ϕ is the marking of F). The map p induces a type preserving map $p_*\colon P_{n,k}(\Sigma_g) \to P_{n,k}(\Sigma_{g+1})$. Consider the map $\mathrm{id} \times p_* \times \mathrm{id}\colon A \times P_{n,k}(\Sigma_g) \times B \to A \times P_{n,k}(\Sigma_{g+1}) \times B$. This map sends $P(\mathbf{N}_i)$ to $P(\mathbf{N}_i')$ and so

$$\mathrm{id} \times p_* \times \mathrm{id}((P(\mathbf{N}_1) \times B) \cap (A \times P(\mathbf{N}_2))) = (P(\mathbf{N}_1') \times B) \cap (A \times P(\mathbf{N}_2')).$$

It is also clear that, near the image of $P(\mathbf{N}_i)$, the set $P(\mathbf{N}_i')$ can be locally described as a product. Any perturbation of $P(\mathbf{N}_1) \times B$ to make it dimensionally transverse to $A \times P(\mathbf{N}_2)$ can be extended trivially in the product directions so that $P(\mathbf{N}_1') \times B$ is dimensionally transverse to $A \times P(\mathbf{N}_2')$ and their intersection remains the image of the intersection. This implies $\pm P(\mathbf{M})$ is independent of the choice of F. $\square$

Remark. The sign ambiguity arise because there are two essentially different choices for the gluing map of a Heegaard splitting of the sphere and they induce morphisms that differ by multiplication by -1.

In defining the homomorphism $P(\mathbf{M})_*$ associated to a framed cobordism $\mathbf{M}$, we worked with spaces of the form $\prod_{i=1}^{r} P_i$ where each of the spaces P_i is of the form $P_{n,k(i)}(\Sigma_{g(i)})$. These spaces are stratified using the product stratification induced by the standard stratifications on the $P_{n,k(i)}(\Sigma_{g(i)})$. Let p be the perversity function on $\prod_{i=1}^{r} P_i$ that assigns middle perversity to a stratum if not all of the strata in the product correspond to representations of the same type and logarithmic perversity if they are all of the same type. Notice that $p \leq s$ where s is the special perversity. Observe further that when the space is of the type $P_{n,k}(\Sigma_g) \times P_{n,k}(\Sigma_g)$ then the perversity p is the same as the "placid perversity" studied in [9]. By [9, Corollary 3.2], the natural map $IH_*^m(P_{n,k}(\Sigma_g) \times P_{n,k}(\Sigma_g)) \to IH_*^p(P_{n,k}(\Sigma_g) \times P_{n,k}(\Sigma_g))$ is an isomorphism. Now if $\mathbf{M}$ is a cobordism constructed by cutting a closed manifold X open along a nonseparating surface F then $P(\mathbf{M})$ lives in some $P_{n,k}(\Sigma_g) \times P_{n,k}(\Sigma_g)$. A more careful analysis of perversities shows that we can choose $P(\mathbf{M})$ so that it is p-allowable. Let $q_{n,k}(t)$ be the polynomial associated to $P(\mathbf{M})$ as in Definition 1.2 and let $\mu_{n,k} = q_{n,k}(1)$. The results of [9] imply that $\mu_{n,k}$ is equal to the intersection number of the induced middle perversity class with a middle perversity class associated to the diagonal of $P_{n,k}(\Sigma_g) \times P_{n,k}(\Sigma_g)$. It follows that if $\mu_{n,k}$ is not zero then the intersection of $P(\mathbf{M})$ with the diagonal must be nonempty. This in turn can be used to show that the intersection of the image of $\mathrm{Rep}(\pi_1(M), PU(n))$ in $P_{n,k}(\Sigma_g) \times P_{n,k}(\Sigma_g)$ with the diagonal is nonempty. These points of intersection correspond to representations of the fundamental group of X into $PU(n)$ that induce a bundle over F having first Chern number congruent to k modulo n. Hence the numerical invariants $\mu_{n,k}$ can be interpreted as an algebraic count of the number of characters of $PU(n)$-representations of the fundamental group of X inducing a prescribed bundle over F. Suppose $K \subset N$ is a homologically trivial knot in a closed oriented rational homology 3-sphere N. Then longitudinal surgery on K yields a closed manifold X and a surface in X corresponding to a "capped off" Seifert surface for K. We define the invariant of the knot K to be the intersection Lefschetz polynomial associated to the cobordism obtained by cutting X along the Seifert surface. Thus the evaluation of this polynomial at

$t = 1$ gives an algebraic count of the number of characters of $PU(n)$-representations of the fundamental group of the result, X, of longitudinal surgery on the knot.

For the purpose of explicit calculation in §5 and §6, it is more convenient to use the spaces $X_{n,k}$ in place of $P_{n,k}$ in order to avoid the need to rework Kirwan's apparatus. We will denote the corresponding polynomial and numerical invariant by $p_{n,k}(t)$ and $\lambda_{n,k}$ respectively. For a knot, the invariant $\lambda_{n,d}$ can be thought of as an algebraic count of the number of characters of $SU(n)$-representations of the fundamental group of the complement of the knot which take a longitude to $e^{2\pi id/n}$ times the identity. In fact, for a fibered knot $q_{n,k}(t) = p_{n,k}(t)$ and consequently $\mu_{n,k} = \lambda_{n,k}$. In this case, $q_{n,k}(t)$ and $p_{n,k}(t)$ are the intersection Lefschetz polynomials of the monodromy action on $X_{n,k}$ and $p_{n,k}$ respectively (see §5). The space $X_{n,k}$ is a finite regular covering space of $P_{n,k}$ with group of covering transformations $G = H_1(\Sigma_g, \mathbb{Z}/n)$. Since G acts trivially on $IH_*^m(X_{n,k})$ (in the case n and k are relatively prime, see [10] or the statement following [2, Proposition 9.7]; the extension to any (n, k) can be deduced from [16]), it follows that the two Lefschetz polynomials of the respective monodromy actions are the same.

§4. FUNCTORIALITY

In this section we prove that the restriction of Q_n to the monotone category $\mathcal{FC}^+_{(2)}$ or $\mathcal{FC}^-_{(2)}$ is a functor. The issue is that if $\mathbf{M}_1$ and $\mathbf{M}_2$ are morphisms in $\mathcal{FC}^+_{(2)}$ or $\mathcal{FC}^-_{(2)}$ then $P(\mathbf{M}_2\mathbf{M}_1)_* = \pm P(\mathbf{M}_2)_* P(\mathbf{M}_1)_*$. We will restrict our attention to $\mathcal{FC}^-_{(2)}$, the other case being similar. Hence throughout this section we will be working with cobordisms so that the "out" surface of any connected component of the cobordism is connected. We begin with an explanation of why this restriction is necessary. The proof of functoriality relies on understanding the intersection of two cycles in a stratified space. We give a criterion for when the intersection homology class carried by an intersection of two cycles is equal to the homological intersection of the two cycles. We then proceed to prove functoriality for some special cases. Assume that $\mathbf{M}_1$ and $\mathbf{M}_2$ are cobordisms whose underlying manifolds are hollow handlebodies. We want to understand the composition $P(\mathbf{M}_2)_* P(\mathbf{M}_1)_*$. There are four cases to consider based on whether the preferred boundary components of $\mathbf{M}_1$ and $\mathbf{M}_2$ are the "in" or "out" components. The most difficult case occurs when the "in" surface of $\mathbf{M}_1$ is preferred and the "out" surface of $\mathbf{M}_2$ is preferred. Once these special cases have been established, the general case follows from formal manipulations.

Recall that in §3 we defined functors Q_n: $\mathcal{FC}' \to Pgr(\mathcal{V}_{\mathbb{Z}})$ where $\mathcal{FC}'$ is one of the monotone categories $\mathcal{FC}^+_{(2)}$ or $\mathcal{FC}^-_{(2)}$. Although $Q_n(\mathbf{M})$ is defined for any framed cobordism $\mathbf{M}$, we require $\mathbf{M}$ to be monotone (as defined in §1) in order to assure that Q_n is a functor. We will illustrate the need for this restriction with the following two abstract examples. Let $\mathbf{M}_1$ and $\mathbf{M}_2$ be a pair of connected cobordisms so that the domain of $\partial_{\mathrm{out}}\mathbf{M}_1$ is the same as the domain of $\partial_{\mathrm{in}}\mathbf{M}_2$. Let M_1 and M_2 be the manifolds underlying $\mathbf{M}_1$ and $\mathbf{M}_2$ respectively.

Example 1. Suppose that the surface underlying $\partial_{\mathrm{out}}\mathbf{M}_1$ is a connected surface. If M_3 is the manifold underlying $\mathbf{M}_3 = \mathbf{M}_2\mathbf{M}_1$ then $\pi_1(M_3) = \pi_1(M_1) \star_{\pi_1(\Sigma_g)} \pi_1(M_2)$ where Σ_g is the domain of the marking of $\partial_{\mathrm{out}}\mathbf{M}_1$ and $\partial_{\mathrm{in}}\mathbf{M}_2$. Let G be an arbitrary group. For any group Γ, let $\mathrm{Rep}(\Gamma, G)$ denote the set of homomorphisms $\Gamma \to G$. Applying the functor $\mathrm{Rep}(\ \ , G)$ to the pushout diagram

$$\begin{array}{ccc} \pi_1(\Sigma_g) & \xrightarrow{\theta} & \pi_1(M_1) \\ \psi\downarrow & & \downarrow \\ \pi_1(M_2) & \longrightarrow & \pi_1(M_3) \end{array}$$

yields the pullback diagram

$$\begin{array}{ccc} \mathrm{Rep}(\pi_1(M_3), G) & \longrightarrow & \mathrm{Rep}(\pi_1(M_2), G) \\ \downarrow & & \psi^* \downarrow \\ \mathrm{Rep}(\pi_1(M_1), G) & \xrightarrow{\theta^*} & \mathrm{Rep}(\pi_1(\Sigma_g), G). \end{array}$$

In particular, $\mathrm{Rep}(\pi(M_3), G)$ is naturally identified with

$$\pi_{13}(\mathrm{graph}\,\theta^* \times \mathrm{Rep}(\pi_1(M_2), G) \cap \mathrm{Rep}(\pi_1(M_1), G) \times \mathrm{graph}'\psi^*)$$

where graph $\theta^* = \{(\rho, \theta^*(\rho))\}$, graph$'$ $\psi^* = \{(\psi^*(\rho), \rho)\}$ and

$$\pi_{13}\colon \mathrm{Rep}(\pi_1(M_1), G) \times \mathrm{Rep}(\pi_1(\Sigma_g), G) \times \mathrm{Rep}(\pi_1(M_2), G) \to \mathrm{Rep}(\pi_1(M_1), G) \times \mathrm{Rep}(\pi_1(M_2), G)$$

is the projection. Hence the cobordism $\mathbf{M}_3$ acts like a correspondence from $\mathrm{Rep}(\pi_1(M_1), G)$ to $\mathrm{Rep}(\pi_1(M_2), G)$ that is the composite of θ^* and the inverse of ψ^* (as a correspondence).

Example 2. Now assume that $\partial_{\mathrm{out}}\mathbf{M}_1$ has a domain that is not connected. The fundamental group of M_3 is no longer the amalgamated free product of the fundamental group of the two pieces as there are additional generators which correspond to arcs joining the connected components of the surface underlying $\partial_{\mathrm{out}}\mathbf{M}_1$. Hence the representations of $\pi_1(M_3)$ are no longer the composition of θ^* and the inverse of ψ^* (as a correspondence); in particular, the composition of representation spaces is not functorial.

In our situation we are working with more complicated spaces of representations and with intersection homology. Nevertheless, the above two examples serve as useful paradigms which motivate the need for restricting the domain of Q_n.

First, we need to establish a lemma about computing intersections in intersection homology. Let X be a compact n-dimensional oriented pseudomanifold with a given stratification. Suppose p and q are loose perversities such that $(X, p + q)$ is permissible. Let $P \subset X$ be an i-dimensional oriented pseudomanifold that is p-allowable and let $Q \subset X$ be an j-dimensional oriented pseudomanifold that is q-allowable. Let $r = t - p - q$ where t is the total perversity. Suppose $S \subset X$ is a $2n - i - j$-dimensional oriented pseudomanifold that is r-allowable. Also assume that $R = P \cap Q$ is an $i + j - n$-dimensional orientable pseudomanifold that is $p + q$-allowable. Let R_S be the locus of points $x \in R$ such that either x lies in a singular stratum of X or x lies in the top stratum of X and P, Q fail to intersect transversely at x.

Proposition 4.1. *If R_S has codimension at least two in R and if R is given the orientation it inherits from $P \cap Q - R_S$ then for all S as above $[S]\cdot[R] = [S]\cdot([P]\cdot[Q])$ where "[ ]" denotes the intersection homology class and "$\cdot$" is the intersection homology intersection product.*

Proof. Let $f\colon P \times [0, 1] \to X$ be a perturbation of P so that $f|_{P\times\{1\}}$ is dimensionally transverse to Q. Assume that S has been perturbed so it that is dimensionally transverse to R, misses R_S and is dimensionally transverse to Q, P and $f|_{P\times\{1\}}$. Applying relative dimensional transversality, perturb f relative to $P\times\{0\}$ and $P\times\{1\}$ so that f is dimensionally transverse to S. Then $f^{-1}(S)$ is a cobordism from $S \cap P$ to $S \cap f(P\times\{1\})$ and so

$$\begin{aligned}[S]\cdot[R] &= ([S]\cdot[P])\cdot[Q] = ([S]\cdot[f|_{P\times\{1\}}])\cdot[Q] \\ &= [S]\cdot([f|_{P\times\{1\}}]\cdot[Q]) = [S]\cdot([P]\cdot[Q]). \quad \square\end{aligned}$$

Now suppose that X, Y and Z are compact oriented pseudomanifolds with only even dimensional strata. Let Λ, M, N be the index sets for their strata. Let $s\colon \Lambda \times M \to \{0, 2, \ldots\}$

and $s: M \times N \to \{0, 2, \ldots\}$ be the special perversity functions (see §2). Define $s_3: \Lambda \times M \times N \to \{0, 2, \ldots\}$ by $s_3(\lambda, \mu, \eta) = s(\lambda, \mu) + s(\mu, \eta)$. Observe that if $P \subset X \times Y \times Z$ is an embedded pseudomanifold that is s_3-allowable then $\pi_{13}(P)$, the projection of P to $X \times Z$, is s-allowable in $X \times Z$. Let $C \subset X \times Y$ and $D \subset Y \times Z$ be oriented s-allowable embedded pseudomanifolds. Suppose that $C \times Z$ and $X \times D$ intersect in a dimensionally transverse fashion away from a locus of points that has least codimension two in their intersection. Suppose further that $(C \times Z) \cap (X \times D)$ is s_3-allowable.

Corollary 4.2. *The homomorphism* $IH_*^m(X; \mathbb{Q}) \to IH_*^m(Z: \mathbb{Q})$ *induced by orienting* $(C \times Z) \cap (X \times D)$ *as the intersection of* $C \times Z$ *and* $X \times D$ *coincides with the composite homomorphism* $D_* C_*$.

Proof. Let $\zeta \in IH_i^m(X; \mathbb{Q})$. Let $\eta \in IH_j^m(X; \mathbb{Q})$ where j is the dimension complementary to $D_* C_*(\zeta)$. If for all ζ, η we have $D_* C_*(\zeta) \cdot \eta = \pi_{13}((C \times Z) \cap (X \times D))_*(\zeta) \cdot \eta$ then Poincaré duality implies that $D_* C_* = \pi_{13}((C \times Z) \cap (X \times D))_*$. Now, with appropriate sign changes, $\pi_{13}((C \times Z) \cap (X \times D))_*(\zeta) \cdot \eta = [(C \times Z) \cap (X \times D)] \cdot (\zeta \times [Y] \times \eta)$. By Proposition 4.1:

$$[(C \times Z) \cap (X \times D)] \cdot (\zeta \times [Y] \times \eta) = [C \times Z] \cdot [X \times D] \cdot (\zeta \times [Y] \times \eta) = D_* C_*(\zeta) \cdot \eta. \quad \square$$

The objects we are working with are smoothly stratified and so we can detect dimensional transversality by checking whether tangent spaces span in a particular stratum. Suppose $C \subset X \times Y$ and $D \subset Y \times Z$ and $(x, y, z) \in X \times Y \times Z$ lies in the top stratum and is a manifold point of both $C \times Z$ and $X \times D$. Then $C \times Z$ and $X \times D$ are transverse at (x, y, z) if and only if $d\pi_Y(T_{(x,y,z)} C \times Z) + d\pi_Y(T_{(x,y,z)} X \times D) = T_y(Y)$ where π_Y: $X \times Y \times Z \to Y$ is projection. Abusing notation, this true if and only if $d\pi_Y(T_{(x,y)} C) + d\pi_Y(T_{(y,z)} D) = T_y(Y)$. Observe that this condition will be satisfied if $d\pi_Y(T_{(x,y)} C) = T_y(Y)$ or $d\pi_Y(T_{(y,z)} D) = T_y(Y)$. Suppose $\mathbf{H}$ is a cobordism whose underlying manifold is a hollow handlebody with the "in" surface as the preferred surface. Let π: $P(\mathbf{H}) \to P(\partial_{\text{out}} \mathbf{H})$ be the projection map. Then the restriction of π, $\pi_|$: $P(\mathbf{H}) \cap X \times Y \to Y$, where X is an stratum of $P(\partial_{\text{in}} \mathbf{H})$ and Y is a stratum of $P(\partial_{\text{out}} \mathbf{H})$ is a submersion at each point of $P(\mathbf{H}) \cap X \times Y$. This follows from the observation that $\pi_1(\mathbf{H})$ is the free product of $\pi_1(\partial_{\text{out}} \mathbf{H})$ with a free group and so one can easily construct paths yielding each tangent vector. Consequently, if $\mathbf{M}$ is any cobordism so that the composition $\mathbf{MH}$ is defined then $P(\mathbf{H}) \times P(\partial_{\text{out}} \mathbf{M})$ is dimensionally transverse to $P(\partial_{\text{in}} \mathbf{H}) \times P(\mathbf{M})$. This leads us to the following proposition:

Proposition 4.3. *If* $\mathbf{H}$ *is a cobordism whose underlying manifold is a hollow handlebody and the preferred surface is either the "in" boundary or the "out" boundary then* $Q_n(\mathbf{H})$ *is induced by* $P(\mathbf{H})$.

Proof. By definition, to construct $Q_n(\mathbf{H})$ we choose a Heegaard splitting of the underlying manifold, $\mathbf{H} = \mathbf{H}_2 \mathbf{H}_1$, and set $Q_n(\mathbf{H}) = P(\mathbf{H}_2)_* P(\mathbf{H}_1)_*$. We can choose a Heegaard splitting so that one of the cobordisms $\mathbf{H}_2$ or $\mathbf{H}_1$ is a product cobordism. It then follows from the discussion above that $P(\mathbf{H}_1) \times P(\partial_{\text{out}} \mathbf{H}_2)$ and $P(\partial_{\text{in}} \mathbf{H}_1) \times P(\mathbf{H}_2)$ intersect in a dimensionally transverse manner in $P(\partial_{\text{in}} \mathbf{H}_1) \times P(\partial_{\text{out}} \mathbf{H}_1) \times P(\partial_{\text{out}} \mathbf{H}_2)$. The projection of this intersection is $P(\mathbf{H})$. Hence $P(\mathbf{H})_* = Q_n(\mathbf{H})$. $\square$

Proposition 4.4. *Suppose that* $\mathbf{N}_1$ *and* $\mathbf{N}_2$ *are cobordisms whose underlying manifolds are hollow handlebodies. Furthermore, suppose that one of the following conditions holds:*

(1) $\partial_{\text{out}} \mathbf{N}_1$ *and* $\partial_{\text{in}} \mathbf{N}_2$ *are preferred*
(2) $\partial_{\text{out}} \mathbf{N}_1$ *and* $\partial_{\text{out}} \mathbf{N}_2$ *are preferred*
(3) $\partial_{\text{in}} \mathbf{N}_1$ *and* $\partial_{\text{in}} \mathbf{N}_2$ *are preferred.*

Then $Q_n(\mathbf{N}_2 \mathbf{N}_1) = Q_n(\mathbf{N}_2) Q_n(\mathbf{N}_1)$.

Proof. By Proposition 4.3, $P(\mathbf{N}_1)_* = Q_n(\mathbf{N}_1)$ and $P(\mathbf{N}_1)_* = Q_n(\mathbf{N}_1)$. The first case then follows immediately from the definition of $Q_n(\mathbf{N}_2\mathbf{N}_1)$ since $\mathbf{N}_1$ and $\mathbf{N}_2$ constitute a Heegaard splitting of $\mathbf{N}_2\mathbf{N}_1$.

Observe that in the second and third cases, $\mathbf{N}_2\mathbf{N}_1$ has a hollow handlebody as its underlying manifold and either $\partial_{\text{in}}\mathbf{N}_2\mathbf{N}_1$ or $\partial_{\text{out}}\mathbf{N}_2\mathbf{N}_1$ is the preferred surface. By Proposition 4.3, $Q_n(\mathbf{N}_2\mathbf{N}_1) = P(\mathbf{N}_2\mathbf{N}_1)_*$. Since $\mathbf{N}_2$ and $\mathbf{N}_1$ have hollow handlebodies as underlying manifolds and either $\partial_{\text{out}}\mathbf{N}_1$ or $\partial_{\text{in}}\mathbf{N}_2$ does not contain the preferred surface, it follows that $P(\partial_{\text{in}}\mathbf{N}_1) \times P(\mathbf{N}_2)$ and $P(\mathbf{N}_1) \times P(\partial_{\text{out}}\mathbf{N}_2)$ intersect in a dimensionally transverse fashion. The projection of this intersection is $P(\mathbf{N}_2\mathbf{N}_1)$. Thus $Q_n(\mathbf{N}_2\mathbf{N}_1) = Q_n(\mathbf{N}_2)Q_n(\mathbf{N}_1)$. $\square$

This leaves us with the most difficult case to deal with. From now on assume that $\mathbf{N}_1$ and $\mathbf{N}_2$ have hollow handlebodies as their underlying manifolds, $\partial_{\text{in}}\mathbf{N}_1$ is preferred, $\partial_{\text{out}}\mathbf{N}_2$ is preferred, $\partial_{\text{in}}\mathbf{N}_1 \neq \emptyset$, $\partial_{\text{out}}\mathbf{N}_1 \neq \emptyset$, $\partial_{\text{out}}\mathbf{N}_2 \neq \emptyset$ and have no components of genus less than 2. In addition, we will assume that $\partial_{\text{out}}\mathbf{N}_1 = \partial_{\text{in}}\mathbf{N}_2$ is connected. This allows us to view $\mathbf{N}_1$ and $\mathbf{N}_2$ as being given by the construction below.

Let H_2 be a 3-manifold that is built by taking surfaces $\{F_i\}_{i=1}^r$ and adding 2-handles to $F_i \times [0, 1]$ along $F_i \times \{1\}$. We assume that the system of curves along which the 2-handles are added are nonseparating, so that the descendant of each $F_i \times \{1\}$ is a surface G_i. Denote the connected component of H_2 containing $F_i \times \{0\}$ and G_i by $H_2(i)$. We also assume that the genus of all G_i and $F_i \times \{0\}$ is greater than two. Make H_2 into a cobordism $\mathbf{N}_2$ by choosing markings $\phi_i\colon \Sigma_{g(i)} \to F_i \times \{0\}$, $\psi_i\colon \Sigma_{g(i)} \to G_i$ and framings k_i for $F_i \times \{0\}$ and G_i. Assume that orientations and markings have been chosen so that $(F_i \times \{0\})_{i=1}^r$ is the surface underlying $\partial_{\text{in}}\mathbf{N}_2$ and $(G_i)_{i=1}^r$ is the surface underlying $\partial_{\text{out}}\mathbf{N}_2$. For each connected component $H_2(i)$ we have $\pi_1(H_2(i)) \cong \pi_1(G_i) \star F(d_i)$ where $F(d_i)$ is a free group of rank equal to the number of 2-handles of H_2 that have their boundary in $F_i \times \{1\}$. Let $S_{n,k_i}(\mathbf{N}_2(i))$ be the space of $PU(n)$ representations of $\pi_1(H_2(i))$ which induce a bundle of Chern number congruent to k_i modulo n on $F_i \times \{0\}$. Since $S_{n,k_i}(\mathbf{N}_2(i)) \cong S_{n,k_i}(\Sigma_{h(i)}) \times PU(n)^{d_i}$ it is a connected, orientable pseudomanifold. Define

$$S_{n,\mathbf{k}}(\mathbf{N}_2(i)) = \prod_{i=1}^{r} S_{n,k_i}(\mathbf{N}_2(i)) \subset \prod_{i=1}^{r} S_{n,k_i}(\Sigma_{g(i)}) \times \prod_{i=1}^{r} S_{n,k_i}(\Sigma_{h(i)}).$$

where $\mathbf{k} = (k_1, \ldots, k_r)$. Observe that the projection of $S_{n,k_i}(\mathbf{N}_2(i))$ to $S_{n,k_i}(\Sigma_{h(i)})$ is a stratified map that is a submersion when restricted to strata.

Now let H_1 be a manifold constructed by adding handles to $G_i \times [0, 1]$ along $G_i \times \{1\}$ so that the descendent of $\bigcup_i G_i \times \{1\}$ is a connected surface K having genus greater than one. We identify $G_i \times \{0\}$ with G_i. Make H_1 into a cobordism $\mathbf{N}_1$ by using the markings $\psi_i\colon \Sigma_{h(i)} \to G_i$ and the framings k_i above and choosing a marking $\sigma\colon \Sigma_h \to K$ and using $k = \sum_{i=1}^r k_i$ as the framing for K so that $(G_i)_{i=1}^r$ is the surface underlying $\partial_{\text{in}}\mathbf{N}_1$ and K is the surface underlying $\partial_{\text{out}}\mathbf{N}_1$. Observe that $\pi_1(H_2) = (\bigstar_{i=1}^r \pi_1(G_i)) \star F(d)$ where $F(d)$ is a free group. Thus, if we let $S_{n,\mathbf{k}}(\mathbf{N}_1(i))$ denote the space of $PU(n)$ representations of $\pi_1(H_1)$ inducing bundles of Chern number congruent to k_i modulo n on G_i, then $S_{n,\mathbf{k}}(\mathbf{N}_1) \cong (\prod_{i=1}^r S_{n,k_i}(\Sigma_{h(i)})) \times PU(n)^d$. Observe that the projection onto $\prod_{i=1}^r S_{n,k_i}(\Sigma_{h(i)})$ is a stratified mapping that is submersive when restricted to strata.

Let $H_3 = H_2 \bigcup_{\cup_i G_i} H_1$. Make H_3 into a framed cobordism $\mathbf{N}_3$ with $\partial_{\text{in}}\mathbf{N}_3 = \partial_{\text{in}}\mathbf{N}_2$ and $\partial_{\text{out}}\mathbf{N}_3 = \partial_{\text{out}}\mathbf{N}_1$, i.e. $\mathbf{N}_3$ is the composite $\mathbf{N}_1\mathbf{N}_2$. Notice that

$$\pi_1(H_3) = (\bigstar_{i=1}^r \pi_1(G_i)) \star (\bigstar_{i=1}^r F(d_i)) \star F(d).$$

Thus the space of $PU(n)$ representations $S_{n,\mathbf{k}}(\mathbf{N}_3)$ (defined similarly to $S_{n,\mathbf{k}}(\mathbf{N}_i)$, $i = 1, 2$) is diffeomorphic to $(\prod_{i=1}^r S_{n,k_i}(G_i)) \times PU(n)^{\Sigma_i d_i + d}$. In particular, $S_{n,\mathbf{k}}(\mathbf{N}_3)$ is connected. Alternatively, $\pi_1(H_3)$ is the amalgamated free product of $\pi_1(H_1)$ with the groups $\pi_1(H_2(i))$ where the groups are amalgamated along $\pi_1(G_i)$ and so $\pi_1(H_3) = \pi_1(H_1) \star \bigstar^r_{i=1\ \pi_1(G_i)} \pi_1(H_2(i))$. Let $C = S_{n,\mathbf{k}}(\mathbf{N}_2) \times S_{n,k}(\Sigma_h)$ and $D = (\prod_{i=1}^r S_{n,k_i}(\Sigma_{g(i)})) \times S_{n,\mathbf{k}}(\mathbf{N}_1)$. Then $S_{n,\mathbf{k}}(\mathbf{N}_3) = C \cap D$. The submersive property of projection from $S_{n,\mathbf{k}}(\mathbf{N}_2)$ to $\prod_{i=1}^r S_{n,k_i}(\Sigma_{h(i)})$ implies that C and D intersect in a dimensionally transverse fashion in the connected set $S_{n,\mathbf{k}}(\mathbf{N}_3)$. This implies that $P(\mathbf{N}_2) \times P_{n,k}(\Sigma_h)$ and $(\prod_{i=1}^r P_{n,k_i}(\Sigma_{g(i)})) \times P(\mathbf{N}_1)$ intersect in a dimensionally transverse fashion in a cycle that projects via π_{13} to a cycle $P'(\mathbf{N}_2) \subset (\prod_{i=1}^r P_{n,k_i}(\Sigma_{g(i)})) \times P_{n,k}(\Sigma_h)$. If Q_n is to be functorial then this cycle should induce the same homomorphism $IH^m_*(\prod_{i=1}^r P_{n,k_i}(\Sigma_{g(i)})) \to IH^m_*(P_{n,k}(\Sigma_h))$ as $P(\mathbf{N}_3)$. In order to verify this, we must construct a cycle inducing $P(\mathbf{N}_3)_*$.

There exists a family of arcs $\{v_i\}$ in H_2 having their endpoints in the G_i so that the result of doing 1-surgery on the G_i along the arcs v_i yields a surface that is parallel to the F_i. There exists a family of arcs $\{w_i\}$ in H_1 having their endpoints in G_i (disjoint from $\{\partial v_i\}$) so that the result of 1-surgery along the arcs w_i on G_i is a surface that is parallel to K. The result of 1-surgery on $\{G_i\}$ along $\{v_i\} \cup \{w_i\}$ is a connected surface F that is a Heegaard surface for $\mathbf{N}_3$. There exists a family of disks $D_i \subset H_3$ with $D_i \cap F = \partial D_i$ and

$$D_i \cap v_j = \begin{cases} \emptyset & \text{if } i \neq j \\ \text{single point of transverse intersection} & \text{if } i = j \end{cases}$$

and $D_i \cap w_j = \emptyset$ for all j. Similarly, there exists a family of disks $E_i \subset H_3$ with $E_i \cap F = \partial E_i$ and

$$E_i \cap w_j = \begin{cases} \emptyset & \text{if } i \neq j \\ \text{single point of transverse intersection} & \text{if } i = j \end{cases}$$

and $E_i \cap v_j = \emptyset$ for all j.

We can also identify $S_{n,\mathbf{k}}(\mathbf{N}_3)$ as the solution set of the following equations. Let δ_i: $S_{n,\mathbf{k}}(F) \to PU(n)$ be the map $\delta_i(\rho) = \rho(\partial D_i)$ and let ε_i: $S_{n,k}(F) \to PU(n)$ be the map $\varepsilon_i(\rho) = \rho(\partial D_i)$ (where basepoints and basepaths have been chosen for ∂D_i and ∂E_i). Then $S_{n,\mathbf{k}}(\mathbf{N}_3)$ can be identified with the set $\{\rho \in S_{n,k}(F) \mid \delta_i(\rho) = \mathrm{id}, \varepsilon_j(\rho) = \mathrm{id} \text{ for all } i\, j\}$.

Let $\{F'_\alpha\}$ be the collection of noncompact connected components of the complement of the union of the curves $\{\partial D_i, \partial E_j\}$. In each F'_α choose an open planar surface P_α so that $F'_\alpha - P_\alpha = F_\alpha$ is a closed surface with one boundary component. Observe that the F_α are in one to one correspondence with the G_i and all have genus greater than one. Also notice that along the solution set of the equations, if ρ restricted to $\pi_1(F'_\alpha)$ is irreducible then ρ restricted to $\pi_1(F_\alpha)$ is irreducible. Finally, notice that if P is the closure of $\cup P_\alpha$ in F then curves ∂D_i and ∂E_j can be made into part of a free basis for $\pi_1(P)$. By Proposition 4.6 (which appears below), the function w: $S_{n,k} \to PU(n)^c$, where c is the number of D_i's plus the number of E_j's, is a submersion along the locus of representations whose restrictions to $\pi_1(F'_\alpha)$ are irreducible. Since the genera of the F_α are all greater than two, the complement of this set has codimension at least two in $S_{n,\mathbf{k}}(\mathbf{N}_3)$ (in fact, typically much larger than two).

Choose a marking τ: $\Sigma_g \to F$ and frame F using $k = \sum_i k_i$. The surface F separates H_3 into two hollow handlebodies. Using the above marking and framing for F, make these two pieces into framed cobordisms $\mathbf{F}_1$ and $\mathbf{F}_2$ so that $\mathbf{N}_3 = \mathbf{F}_2\mathbf{F}_1$. By Proposition 4.6, $P(\mathbf{F}_1) \times P_{n,k}(\Sigma_h)$ and $(\prod_{i=1}^r P_{n,k_i}(\Sigma_{g(i)})) \times P(\mathbf{F}_2)$ intersect transversely in top stratum away from a locus having codimension at least two in the intersection where the intersection is contained in $(\prod_{i=1}^r P_{n,k_i}(\Sigma_{g(i)})) \times P_{n,k}(\Sigma_g) \times P_{n,k}(\Sigma_h)$. Hence, up to sign, $\pi_{13}((P(\mathbf{F}_1) \times P_{n,k}(\Sigma_h)) \cap ((\prod_{i=1}^r P_{n,k_i}(\Sigma_{g(i)})) \times P(\mathbf{F}_2))$ induces the same homomorphism as $P(\mathbf{N}_3)$. The image $\pm\,\pi_{13}((P(\mathbf{F}_1) \times P_{n,k}(\Sigma_h)) \cap ((\prod_{i=1}^r P_{n,k_i}(\Sigma_{g(i)})) \times P(\mathbf{F}_2))$ is the same as the image $\pm\,\pi_{13}\ ((P(\mathbf{N}_2) \times P_{n,k}(\Sigma_h)) \cap ((\prod_{i=1}^r P_{n,k_i}(\Sigma_{g(i)})) \times P(\mathbf{N}_1))$ as they can both be identified with the projection of $S_{n,k}(\mathbf{N}_3)$. Furthermore, π_{13} in both cases is one to one when restricted to open dense subsets. In the first case π_{13} is one to one when restricted to those representations whose restriction to $\pi_1(F'_\alpha)$ are irreducible and in the second case when restricted to those representations whose restriction to $\pi_1(G_i)$ are irreducible. Since the intersection is connected the signs either agree or disagree everywhere. Using Proposition 2.3, we see that the admissibility hypotheses of Corollary 4.2 are satisfied. Applying Corollary 4.2 proves:

PROPOSITION 4.5. *Suppose that* $\mathbf{N}_1$, $\mathbf{N}_2$ and $\mathbf{N}_3$ *are as above. Then*

$$P(\mathbf{N}_1)_* P(\mathbf{N}_2)_* = P(\mathbf{N}_3)_*. \quad \square$$

Next, we prove Proposition 4.6 (already used above). Let F be a closed surface with a given basepoint and $\{a_i, b_i\}$ a standard generating set for $\pi_1(F)$ so that $\pi_1(F) = \langle a_i, b_i \mid \prod_{i=1}^{g} [a_i, b_i] = 1 \rangle$. Let w: $PU(n)^{2g} \to PU(n)$ be the "word map" given by $w(A_1, \ldots, A_g, B_1, \ldots, B_g) = \prod_{i=1}^{g} [A_i, B_i]$. Then $\mathrm{Rep}(\pi_1(F), PU(n))$ is naturally identified with $w^{-1}(\mathrm{Id})$. It is a straightforward computation to verify that $(A_1, \ldots, A_g, B_1, \ldots, B_g)$ is a regular point of w if and only if the corresponding representation of the free group on $2g$ letters is irreducible. Provided the genus of F is greater than one, the set of points of $\mathrm{Rep}(\pi_1(F), PU(n))$ that are *not* regular points of w has codimension at least two in $\mathrm{Rep}(\pi_1(F), PU(n))$.

Let $\{\gamma_i\}$ be a system of closed curves on the surface F. Suppose there exists a connected proper subsurface $P \subset F$ so that the closure of $F - P$ is a disjoint union of connected surfaces, F_α, such that each F_α has single boundary component; furthermore, suppose $\{\gamma_i\}$ is part of a free basis for $\pi_1(P)$ (after choosing basearcs to the basepoint).

PROPOSITION 4.6. *Let* F, $\{\gamma_i \mid i = 1, \ldots, r\}$, P *and* $\{F_\alpha \mid \alpha \in A\}$ *be as above. Let* v: $S_{n,k}(F) \to PU(n)^r$ *be the map* $v(\rho) = (\rho(\gamma_1), \ldots, \rho(\gamma_r))$ *and suppose that* ρ_0: $\pi_1(F) \to PU(n)$ *is a point of* $S_{n,k}(F)$ *such that for all* α *the restriction of* ρ_0 *to* $\pi_1(F_\alpha)$ *is irreducible. Then* v *is a submersion at* ρ_0.

Proof. Choose a free basis $\{C_i\}$ for $\pi_1(P)$ so that $\{\gamma_i\}$ is a subset. Choose any smooth paths X_i: $(-\varepsilon, \varepsilon) \to PU(n)$ so that $X_i(0) = \rho_0(C_i)$. Define ∂: $\mathrm{Rep}(\pi_1(P), PU(n)) \to PU(n)^a$ by $\partial\rho = (\rho(\alpha))_{\alpha \in A}$ where a is the number of elements in A and α is represented by the boundary component of F_α. Since the restriction of ρ_0 to $\pi_1(F_\alpha)$ is a regular point of w: $S_{n,k}(F_\alpha) \to PU(n)$ for all α we can find paths $\rho_\alpha(t)$: $\pi_1(F_\alpha) \to PU(n)$ so that $w\rho_\alpha(t) = (\partial\rho_0)_\alpha X_i(t)$ for t near 0. Using these paths, construct a path ρ_t: $\pi_1(F) \to PU(n)$ such that $\rho_t(\gamma_i) = X_i(t)$ for small t. This shows v is submersive at ρ_0. $\square$

We are now able to prove the main theorem of this section.

THEOREM 4.7 (Functoriality). *Let* $\mathscr{FC}'$ *be one of the monotone subcategories* $\mathscr{FC}^{+}_{(2)}$ *or* $\mathscr{FC}^{-}_{(2)}$. *Then* Q_n: $\mathscr{FC}' \to Pgr(\mathscr{V}_{\mathbb{Z}})$ *is a functor.*

Proof. We will treat the case $\mathscr{FC}' = \mathscr{FC}^{-}_{(2)}$, the other case being similar. Assume $\mathbf{M}_1$ and $\mathbf{M}_2$ are monotone decreasing; furthermore, assume $\mathbf{M}_2$ is connected as the disconnected case can be derived by taking the tensor product over the various components. We can assume that we have Heegaard decompositions $\mathbf{M}_1 = \mathbf{N}_2\mathbf{N}_1$ and $\mathbf{M}_2 = \mathbf{F}_2\mathbf{F}_1$; furthermore, assume $\mathbf{F}_1$ is connected and $\mathbf{N}_2$ is monotone decreasing. Then $\pm P(\mathbf{M}_1)_* = \pm P(\mathbf{N}_2)_* P(\mathbf{N}_1)_*$ and $\pm P(\mathbf{M}_2)_* = \pm P(\mathbf{F}_2)_* P(\mathbf{F}_1)_*$. Thus if $\mathbf{M}_1$ and $\mathbf{M}_2$ are composable then $\pm P(\mathbf{M}_2)_* P(\mathbf{M}_1)_* = \pm P(\mathbf{F}_2)_* P(\mathbf{F}_1)_* P(\mathbf{N}_2)_* P(\mathbf{N}_1)_*$. By Proposition 4.5, $\pm P(\mathbf{F}_1)_* P(\mathbf{N}_2)_* = \pm P(\mathbf{F}_1\mathbf{N}_2)_*$. If $\mathbf{K}_1$ and $\mathbf{K}_2$ are cobordisms arising from a Heegaard splitting of $\mathbf{F}_1\mathbf{N}_2$ then, by definition, $\pm P(\mathbf{F}_1\mathbf{N}_2)_* = \pm P(\mathbf{K}_2)_* P(\mathbf{K}_1)_*$. Thus

$\pm P(\mathbf{M}_2)_* P(\mathbf{M}_1)_* = \pm P(\mathbf{F}_2)_* P(\mathbf{K}_2)_* P(\mathbf{K}_1)_* P(\mathbf{N}_1)_*$. The cobordisms $\mathbf{F}_2$, $\mathbf{K}_2$, $\mathbf{K}_1$ and $\mathbf{N}_1$ are obtained from Heegaard splittings of $\mathbf{F}_2\mathbf{K}_2$ and $\mathbf{K}_1\mathbf{N}_1$ and so $\pm P(\mathbf{M}_2)_* P(\mathbf{M}_1)_* = \pm P(\mathbf{F}_2\mathbf{K}_2)_* P(\mathbf{K}_1\mathbf{N}_1)_*$. Note that by Proposition 4.3, the homomorphism associated to a Heegaard splitting of a hollow handlebody agrees with the homomorphism obtained using the character variety. Since $\mathbf{F}_2\mathbf{K}_2$ and $\mathbf{K}_1\mathbf{N}_1$ are cobordisms arising from a Heegaard splitting of $\mathbf{M}_2\mathbf{M}_1$ we have, by definition, $\pm P(\mathbf{M}_2\mathbf{M}_1)_* = \pm P(\mathbf{F}_2\mathbf{K}_2)_* P(\mathbf{K}_1\mathbf{N}_1)_*$. Thus $\pm P(\mathbf{M}_2\mathbf{M}_1)_* = \pm P(\mathbf{M}_2)_* P(\mathbf{M}_1)_*$. $\square$

§5. COMPUTATION OF THE INVARIANTS

Let $K \subset M^3$ be a homologically trivial knot inside a closed oriented 3-manifold. Recall that K is said to be *fibered* if the knot complement, $M^3 - N(K)$, where $N(K)$ is the interior of a tubular neighborhood of K, has the structure of a fiber bundle over the circle and the fiber is a compact orientable surface with one boundary component. The goal of this section is to give an algorithmic method of computing the invariants $\lambda_{n,d}$ and $p_{n,d}(t)$ of §3 in the special case of a fibered knot. The main results are Theorems 5.21 and 5.22.

In §5(A) we show how the Alexander polynomial of an arbitrary homologically trivial knot in a closed oriented 3-manifold can be realized as the Lefschetz polynomial of a certain correspondence associated to a free Seifert surface for the knot; see Proposition 5.4. This result can be regarded as the computation of the "rank one" or "abelian" case of our theory (also see [6]). In the general "non-abelian" case we are concerned with the moduli space of semistable holomorphic bundles of rank n and degree d and fixed determinant over a compact Riemann surface. When n and d are *not* relatively prime, the situation of most interest to us here, this moduli space is typically singular. Our method of calculation relies heavily on the theory developed by Frances Kirwan ([14, 15, 16, 17]), for desingularizing these spaces and computing their (mid-perversity) intersection homology. The required adaptation of Kirwan's technical apparatus to the computation of intersection homology Lefschetz polynomials is described in §5(B). In §5(C), we apply the invariant theory of the symmetric group to the computation of Lefschetz series; see Proposition 5.15 and Theorem 5.17. The final subsection, §5(D), is devoted to the proofs of Theorems 5.21 and 5.22.

(A) *The Alexander polynomial as a Lefschetz polynomial*

We first recall some elementary linear algebra. Let V be a n-dimensional vector space, $A\colon V \to V$ an endomorphism. $\bigwedge^r V$ will denote the r-th exterior power of V and $\bigwedge^r A\colon \bigwedge^r V \to \bigwedge^r V$ the endomorphism induced by A. The characteristic polynomial of A is given by the familiar formula:

$$\det(tI - A) = \sum_{k=0}^{n} (-1)^{n-k} \operatorname{trace}(\textstyle\bigwedge^{n-k} A)\, t^k. \tag{5.1}$$

Now suppose $(V, \langle\,,\rangle)$ is an oriented real n-dimensional inner product space. The inner product on V induces an unique inner product on $\bigwedge^k V$ with the property $\langle w_1 \wedge \dots \wedge w_k, v_1 \wedge \dots \wedge v_k\rangle = \det(\langle w_i, v_j\rangle)$. Let ω be a volume form for V, i.e. ω is an n-form such that $\|\omega\| = 1$ and ω determines the orientation of V. The Hodge star is a linear map $\star\colon \bigwedge^r V \to \bigwedge^{n-r} V$ defined by the condition $\eta \wedge \star\eta = \|\eta\|^2\omega$. Recall that $\star\star = (-1)^{r(n-r)}$ and $\langle x, y\rangle = \star(x \wedge \star y)$ for $x, y \in \bigwedge^r V$.

LEMMA 5.2. *Suppose $A\colon V \to V$ is invertible. Then*

$$\det(A) \textstyle\bigwedge^{n-r}(A^{-1}) = (-1)^{r(n-r)} \star (\bigwedge^r A^*) \star$$

where A^* is the adjoint of A.

Proof. We have $(\bigwedge^r A)(x) \wedge (\bigwedge^{n-r} A)(y) = (\bigwedge^n A)(x \wedge y) = \det(A) x \wedge y$. Thus for any $w, v \in \bigwedge^r V$ $(\bigwedge^r A)(w) \wedge \star\star(\bigwedge^{n-r} A)(\star v) = (-1)^{r(n-r)} \det(A) w \wedge \star v$ or, equivalently, $\langle (\bigwedge^r A)(w), \star(\bigwedge^{n-r} A)(\star v)\rangle = (-1)^{r(n-r)} \det(A) \langle w, v\rangle$. This implies $\bigwedge^r A^* \star (\bigwedge^{n-r} A)\star = (-1)^{r(n-r)} \det(A) I$ from which the conclusion of the lemma follows. $\square$

PROPOSITION 5.3. *Let $(V, \langle\,,\rangle)$ be as above, A, B: $V \to V$ endomorphisms and t be an indeterminate. Then*

$$\det(tA - B) = \sum_{k=0}^{n} (-1)^{(k+1)n} \operatorname{trace}(\star \bigwedge^{n-k} B^* \star \bigwedge^k A) t^k.$$

Proof. The coefficients of t^k on both sides of the asserted identity are clearly continuous functions of the endomorphisms A, B and so it sufficient to prove the identity in the case A is invertible. Using (5.1) and Lemma 5.2 we have:

$$\begin{aligned}
\det(tA - B) &= \det(tA^* - B^*) = \det(A^*) \det(tI - (A^*)^{-1} B^*) t^k \\
&= \det(A^*) \sum_{k=0}^{n} (-1)^{n-k} \operatorname{trace}(\textstyle\bigwedge^{n-k}((A^*)^{-1} B^*)) t^k \\
&= \sum_{k=0}^{n} (-1)^{n-k} \operatorname{trace}(\det(A^*) \textstyle\bigwedge^{n-k}((A^*)^{-1}) \bigwedge^{n-k} B^*) t^k \\
&= \sum_{k=0}^{n} (-1)^{n-k} (-1)^{k(n-k)} \operatorname{trace}(\star \textstyle\bigwedge^k A \star \wedge^{n-k} B^*) t^k \\
&= \sum_{k=0}^{n} (-1)^{(k+1)n} \operatorname{trace}(\star \textstyle\bigwedge^{n-k} B^* \star \bigwedge^k A). \quad \square
\end{aligned}$$

Let M^3 be a compact connected 3-manifold (not necessarily a rational homology sphere) and $K \subset M$ a homologically trivial knot with a given Seifert surface F and basis for $H_1(F)$. This data determines a Seifert matrix V_K for K and we define the *unnormalized Alexander polynomial of* K to be $\det(V_K^T - tV_K)$ where V_K^T is the transpose of V_K. When $M = S^3$ this polynomial coincides with the polynomial defined using the Alexander module, see [20, 8.C.4].

If Y is a topological space with a basepoint let $J(Y)$ denote the space of representations of the fundamental group of Y to $U(1)$. Suppose we are given a knot $K \subset M^3$ with a free Seifert surface F with a bicollaring $F \times I \subset M$ so that $H = \overline{M - F \times I}$ is a handlebody with boundary $S = \partial F \times I = F_0 \cup F_1$ and $K = F_0 \cap F_1$. Choose a common basepoint on K. If the genus of F is g then $J(F_i)$, $i = 0, 1$, and $J(H)$ are compact tori of dimension $2g$ giving rise to correspondences i: $J(H) \subsetneq J(F_0) \times J(F_1)$ and j: $J(F \times I) \subsetneq J(F_0) \times J(F_1)$.

PROPOSITION 5.4. *The Lefschetz polynomial, $L_t(i, j)$, of the pair of correspondences (i, j) is the (un-normalized) Alexander polynomial of the knot K.*

Proof. Suppose $B = \{x_1, \ldots, x_{2g}\}$ is a basis for $H_1(F)$ which is represented geometrically by u_i: $S^1 \to F$ (fix a standard orientation on S^1). For $k = 0, 1$, the inclusion i_k: $F_k \subsetneq F \times I$ induces an isomorphism $H_1(F_k) \cong H_1(F)$ and hence determines a basis $B^k = \{x_1^k, \ldots, x_{2g}^k\}$ for $H_1(F_k)$. The u_i's induce $J(F \times I) \to J(S^1) = S^1$ which determine a basis for $H^1(J(F \times I))$ and hence a basis $\{X_1^0, \ldots, X_{2g}^0\}$ for $H^1(J(F_0)) \cong H^1(J(F \times I))$. The linear map $H_1(F_0) \to H^1(J(F_0))$ given by $x_i^0 \mapsto X_i^0$ is an isomorphism. Similarly, B^1 determines a basis for $H_1(H)$ via the isomorphism $H_1(F_1) \cong H_1(H)$ induced by inclusion

$F_1 \subset H$ and thus a basis for $H^1(J(H))$. There is a commutative diagram:

$$\begin{array}{ccccc} H^1(J(F_0)) & \xrightarrow{\pi_0^*} & H^1(J(F_0)\times J(F_1)) & \xrightarrow{i^*} & H^1(J(H)) \\ \cong\uparrow & & \cong\uparrow & & \cong\uparrow \\ H_1(F_0) & \xrightarrow{(i_0)_*} & H_1(F_0\cup F_1) & \xrightarrow{i_*} & H_1(H) \end{array}$$

where π_k: $J(F_0)\times J(F_1)\to J(F_k)$, $k = 0, 1$, is projection. The matrix, denoted V_K, of $i_*(i_0)_*$ with respect to the given bases is the Seifert matrix of the knot K and the map $\phi^1 \equiv i^*\pi_0^*$ is represented by the same matrix. For $0 \le k \le 2g$ define ϕ^k to be the composite:

$$H^k(J(F_0))\xrightarrow{\pi_0^*} H^k(J(F_0)\times J(F_1))\xrightarrow{i^*} H^k(J(H))$$

and ϕ_k to be the composite:

$$H_k(J(H))\xrightarrow{i_*} H_k(J(F_0)\times J(F_1))\xrightarrow{(\pi_i)_*} H^k(J(F_1)).$$

$H^*(J(F_0))$ and $H^*(J(H))$ are algebras via the cup product and there are natural isomorphisms of algebras $\bigwedge^* H^1(J(F_0)) \cong H^*(J(F_0))$ and $\bigwedge^* H^1(J(H)) \cong H^*(J(H))$. Also, $H_*(J(F_1))$ and $H_*(J(H))$ are algebras via the Pontryagin product and there are natural isomorphisms of algebras $\bigwedge^* H_1(J(F_1)) \cong H_*(J(F_0))$ and $\bigwedge^* H_1(J(H)) \cong H_*(J(H))$. Now, $\phi^k = \bigwedge^k \phi^1$ and $\phi_k = \bigwedge^k \phi_1$; furthermore, ϕ^1 is the transpose (Hom-dual) of ϕ_1 (using the identification $H_1(F_0) \cong H_1(F\times I) \cong H_1(F_1)$). The Lefschetz polynomial of the pair of correspondences (i,j) is defined by:

$$L_t(i,j) = \sum_{k=0}^{2g} (-1)^k \operatorname{trace}(\mu_k(D^{-1}\phi_{2g-k}D)\phi^k)t^k$$

where D: $H^k(J(H))\to H_{2g-k}(J(H))$ and D^{-1}: $H_{2g-k}(J(F_1))\to H^k(J(F_1))$ are respectively the Poincaré duality isomorphism and its inverse. The map μ_k: $H^k(J(F_1))\to H^k(J(F_0))$ is the composite of the six homomorphisms:

$$H^k(J(F_1))\xrightarrow{\pi_1^*} H^k(J(F_0)\times J(F_1))\xrightarrow{j^*} H^k(J(F\times I))\xrightarrow{D} H_{2g-k}(J(F\times I))$$

$$\xrightarrow{j_*} H_{2g-k}(J(F_0)\times J(F_1))\xrightarrow{(\pi_0)_*} H_{2g-k}(J(F_0))\xrightarrow{D^{-1}} H^k(J(F_0)).$$

The matrix of μ_k with respect to the given bases is $(-1)^{k(2g-k)}$ times the identity. Applying Proposition 5.3, we have $L_t(i,j) = \det(tV_K - V_K^T) = (-1)^{2g}\det(V_K^T - tV_K)$. □

(B) An extension of Kirwan's theory

The character varieties of §2 have alternate descriptions in terms of holomorphic bundles over a Riemann surface.

Let $\mathscr{M}(n, d)$ be the moduli space of semi-stable holomorphic vector bundles of rank n and degree d over a compact Riemann surface F of genus g. When n and d are relatively prime $\mathscr{M}(n, d)$ is a smooth projective variety whose rational cohomology has been computed by Atiyah and Bott, [2], and by Harder and Narasimhan, [10]. When n and d are not relatively prime, the case of most interest to us, $\mathscr{M}(n, d)$ is a singular projective variety provided $g \geq 3$ or $g = 2$ and $(n, d) \neq (2, 2k)$. The intersection Poincaré polynomial of $\mathcal{M}(n, d)$, $IP_t(\mathcal{M}(n, d)) \equiv \sum_i (-1)^i \operatorname{rank} IH_i^m(\mathcal{M}(n, d))t^i$, was computed by Kirwan in [16] based on her general theory developed in the papers [14], [15], and [17]. We briefly review her calculation below and indicate the minor modifications needed to generalize it to an algorithm for the computation of the intersection Lefschetz polynomial of a homeomorphism $f\colon \mathcal{M}(n, d) \to \mathcal{M}(n, d)$ induced by an orientation preserving self-homeomorphism of the underlying surface.

The variety $\mathcal{M}(n, d)$ has several descriptions as a quotient in the sense of geometric invariant theory ("invariant quotient"). Let F be the underlying Riemann surface, $G(n, p)$ the complex Grassmannian of n-dimensional quotients of $\mathbb{C}^p$, and T the tautological bundle over $G(n, p)$. Let $R(n, d)$ be the set of holomorphic maps $h\colon F \to G(n, p)$ such that the bundle $E(h) = h^*T$ has degree d and the map on sections $\mathbb{C}^p \to H^0(F, E(h))$ induced from the quotient bundle map $\mathbb{C}^p \times F \to E(h)$ is an isomorphism. For sufficiently large d, $R(n, d)$ is a quasiprojective variety. There is a natural action of the general linear group $GL(p)$ on $R(n, d)$ and the invariant quotient $R(n, d)//SL(p)$ can be taken as one definition of $\mathcal{M}(n, d)$. Note that $\mathcal{M}(n, d) \cong \mathcal{M}(n, d + kn)$ for $k \in \mathbb{Z}$; the isomorphism is given by tensoring with a line bundle of degree k. Kirwan provides an algorithm ([16] based on [15]) for equivariantly blowing up $R(n, d)$ along smooth subvarieties so as to obtain $\tilde{R}(n, d)$ whose invariant quotient, $\tilde{\mathcal{M}}(n, d) = \tilde{R}(n, d)//SL(p)$, is a partial desingularization of $\mathcal{M}(n, d)$ in the sense that, locally, $\tilde{\mathcal{M}}(n, d)$ is the quotient of a nonsingular variety by finite group action. In particular $\tilde{\mathcal{M}}(n, d)$ is a rational cohomology manifold.

The open subset of semi-stable points of a reductive group action on a variety V will be denoted V^{ss}. Let $\mathcal{R}(k)$ be a set of representatives of conjugacy classes of all connected reductive subgroups R of complex dimension k in $GL(p)$ such that $Z_R^{ss} = \{x \in R(n, d)^{ss} \mid R \text{ fixes } x\}$ is nonempty. $\tilde{Z}_R^{ss}$ will denote the proper transform of Z_R^{ss} in $\tilde{R}(n, d)^{ss}$. Let $R \in \mathcal{R}(k)$ which we can take to be of the form $R = \prod_{1 \leq j \leq s} GL(m_j)$ where $k = m_1^2 + \cdots + m_s^2$. For $h \in Z_R^{ss}$, let E be the holomorphic bundle $E = E(h)$ and write $E = m_1 E_1 \oplus \cdots \oplus m_s E_s$ with each E_j stable of rank n_j and degree $d_j = n_j d/n$ and $E_i \neq E_j$ if $i \neq j$. Let $\operatorname{End}'_\oplus E$ be the quotient of the bundle of holomorphic endomorphisms of E by the subbundle consisting of those endomorphisms which preserve the direct sum decomposition of E. The normal to the orbit of $\tilde{Z}_R^{ss}$ in $\tilde{R}(n, d)^{ss}$ at the point $h \in Z_R^{ss}$ is naturally identified with $H^1(F, \operatorname{End}'_\oplus E)$. Let $\mathcal{B}(R)$ be the set of all *β-sequences* (see [16, 3.18] and [14, 5.11]) for the normal representation ρ of R on $H^1(F, \operatorname{End}'_\oplus E)$. Associated to a beta sequence $\beta \in \mathcal{B}(R)$ there is a stabilizer $\operatorname{Stab}(\beta)$, a reductive subgroup of R, and non-negative integers $z(\beta)$, $d(\beta)$, and $q(\beta)$ calculated from the weights of ρ and the data constituting β ([16, 3.18]). Also, if $\prod_{1 \leq j \leq s} Y_j$ is any product of quasiprojective varieties Y_j and $\{1, \ldots, s\}$ is the disjoint union of nonempty subsets $I_1, \ldots, I_v$ such that for $i \in I_p$ and $j \in I_q$ $Y_i = Y_j$ if and only if $p = q$, define $[\prod Y_j]$ by

$$\left[\prod Y_j\right] = \prod_{1 \leq p \leq v} \left[\sum_{i \in I_p} Y_i\right]$$

where on the right $[\prod_{i \in I_p} Y_i]$ is the product $\prod_{i \in I_p} Y_i$ blownup along all diagonals (see [16, 3.9]). We can now state Kirwan's theorems:

PROPOSITION 5.5 ([16, 3.20]). *The Poincaré polynomial of the blownup moduli space is given by* $P_t(\tilde{\mathcal{M}}(n, d)) = (1 - t^2) P_t^{GL(p)}(\tilde{R}(n, d)^{ss})$ *where*

$$P_t^{GL(p)}(\tilde{R}(n,d)^{ss}) = P_t^{GL(p)}(R(n,d)^{ss}) - \sum_{1 \le k \le n^2} \sum_{R \in \mathscr{R}(k)} P_t^N(\tilde{Z}_R^{ss})$$

$$+ \sum_{1 \le k \le n^2} \sum_{R \in \mathscr{R}(k)} \sum_{\beta \in \mathscr{B}(R)} (-1)^{q(\beta)} w(\beta)^{-1} t^{2d(\beta)}(1-t^2)^{-1}(1-t^{2z(\beta)+2}) P_t^{N \cap \mathrm{Stab}(\beta)}(\tilde{Z}_R^{ss})$$

where N is the normalizer of R in $GL(p)$. Let N_0 be the identity component of N. Then for $R = \prod_{1 \le j \le s} GL(m_j)$, $H^*_{N \cap \mathrm{Stab}(\beta)}(\tilde{Z}_R^{ss})$ is the invariant part of

$$H^*(B(R \cap \mathrm{Stab}(\beta))) \otimes H^*(\tilde{Z}_R^{ss}//N_0)$$

$$= \bigotimes_{1 \le j \le s} H^*(B(GL(m_j) \cap \mathrm{Stab}(\beta))) \otimes H^*\left(\left[\prod_{1 \le j \le s} \tilde{\mathscr{M}}(n_j, d_j)\right]\right)$$

under the natural action of $\pi_0(R(N \cap \mathrm{Stab}(\beta))) \subset \pi_0 N$ by permuting factors.

The intersection Poincaré polynomial of $\mathscr{M}(n,d)$ is obtained by subtracting an appropriate correction term, involving the intersection homology of the links of the singular strata, from the Poincaré polynomial of $\tilde{\mathscr{M}}(n,d)$. Let $t(s)$ be the integer function (see [16, p. 253] or [17, 2.1]) defined by

$$t(s) = \begin{cases} s-2 & \text{if } s < \dim \mathbb{P}(H^1(F, \mathrm{End}'_{\oplus} E))//R \\ s & \text{otherwise} \end{cases}$$

where $\mathbb{P}(H^1(F, \mathrm{End}'_{\oplus} E))$ is the projectivization of the complex vector space $H^1(F, \mathrm{End}'_{\oplus} E)$. We will use the notation $\mathrm{Inv}_H V$ for the invariant part of a group action H on a vector space V. Let $\mathscr{R} = \bigcup_{1 \le k \le n^2} \mathscr{R}(k)$.

Proposition 5.6 ([16, 3.21]). *For $R = \prod_{1 \le j \le s} GL(m_j) \in \mathscr{R}$ define $S(R, q, q', q'')$ to be*

$$H^q\left(\left[\prod_{1 \le j \le s} \tilde{\mathscr{M}}(n_j, d_j)\right]\right) \otimes H^{q'}\left(\prod_{1 \le j \le s} BGL(m_j)\right) \otimes IH^m_{t(q'')}(\mathbb{P}(H^1(F, \mathrm{End}'_{\oplus} E))//R).$$

Then the intersection Poincaré polynomial of $\mathscr{M}(n,d)$ is given by:

$$IP_t(\mathscr{M}(n,d)) = P_t(\tilde{\mathscr{M}}(n,d)) - \sum_{i \ge 0} \sum_{R \in \mathscr{R}} \sum_{q+q'+q''=i} (-1)^i \dim \mathrm{Inv}_{\pi_0 N} S(R, q, q', q'') t^i$$

where N is the normalizer of R in $GL(p)$.

Remark. Kirwan uses nonalternating Poincaré polynomials and series; however, the formula for the Poincaré polynomial of $\tilde{\mathscr{M}}(n,d)$ takes the same form with either convention.

For a given compact Riemann surface F the complex structure of $R(n,d)$, $\mathscr{M}(n,d)$, $\tilde{R}(n,d)$, and $\tilde{\mathscr{M}}(n,d)$ is determined by the complex structure of F. When we wish to emphasize this dependence, we will write $R(n,d)_F$, $\mathscr{M}(n,d)_F$, $\tilde{R}(n,d)_F$, and $\mathscr{M}(n,d)_F$. Let $\mathscr{T}_F$ be the Teichmüller space associated to F. A point in $\mathscr{T}_F$ is the equivalence class of a pair (S,f) where S is a Riemann surface and $f: S \to F$ is a quasiconformal homeomorphism. $\mathscr{T}_F$ is homeomorphic to $\mathbb{R}^{6g-6}$, where g is the genus of F, and has a preferred basepoint $[(F, 1_F)] \in \mathscr{T}$ where 1_F the identity $1_F: F \to F$. We can form a bundle $\mathscr{T}\mathscr{M}(n,d) \to \mathscr{T}_F$ whose fiber over $[(S,f)] \in \mathscr{T}_F$ is biholomorphic to $\mathscr{M}(n,d)_S$; indeed, for a natural complex structure on $\mathscr{T}_F$ it has the structure of a holomorphic bundle (see [11]). Similarly, there is a $GL(p)$ equivariant bundle $\mathscr{T}R(n,d) \to \mathscr{T}_F$ whose fiber over $[(S,f)] \in \mathscr{T}_F$ is biholomorphic to $R(n,d)_S$. Kirwan's algorithm for blowing up $R(n,d)$ can be applied fiberwise to obtain a $GL(p)$ equivariant bundle $\mathscr{T}\tilde{R}(n,d) \to \mathscr{T}_F$ whose fiber over $[(S,f)]$ is biholomorphic to $\tilde{R}(n,d)_S$. Passing to the invariant quotient, we obtain a bundle $\mathscr{T}\tilde{\mathscr{M}}(n,d) \to \mathscr{T}_F$

whose fiber over $[(S,f)]$ is biholomorphic to $\tilde{\mathcal{M}}(n,d)_S$; furthermore, the bundle projection factors through $\mathcal{T}\mathcal{M}(n,d) \to \mathcal{T}_F$ and $\mathcal{T}\tilde{\mathcal{M}}(n,d)$ is a partial desingularization of $\mathcal{T}\mathcal{M}(n,d)$. Similarly, for any reductive subgroup R of $GL(p)$ with normalizer N we obtain bundles $\mathcal{T}Z_R^{ss}//N \to \mathcal{T}_F$ and $\mathcal{T}\tilde{Z}_R^{ss}//N \to \mathcal{T}_F$.

Let $h: F \to F$ be an orientation preserving homeomorphism. Then h can be represented in its homotopy class by a quasiconformal homeomorphism which will also be denoted by h. There is an induced biholomorphism $\mathcal{T}_F \to \mathcal{T}_F$ on Teichmüller space given by $[(S,f)] \mapsto [(S,hf)]$. This map is covered by holomorphic maps $\mathcal{T}R(n,d) \to \mathcal{T}R(n,d)$, $\mathcal{T}M(n,d) \to \mathcal{T}M(n,d)$, and $\mathcal{T}Z_R^{ss}//N \to \mathcal{T}Z_R^{ss}//N$ all of which will be denoted by h as it will be clear from the context which is meant. These maps extend to the blownup objects yielding bundle maps over $\mathcal{T}_F \to \mathcal{T}_F$: $\mathcal{T}\tilde{R}(n,d) \to \mathcal{T}\tilde{R}(n,d)$, $\mathcal{T}\tilde{\mathcal{M}}(n,d) \to \mathcal{T}\tilde{\mathcal{M}}(n,d)$, and $\mathcal{T}\tilde{Z}_R^{ss}//N \to \mathcal{T}\tilde{Z}_R^{ss}//N$ all of which will be denoted by $\tilde{h}$.

Since $\mathcal{T}_F$ is contractible, we can choose a topological trivialization of $\mathcal{T}\tilde{R}(n,d) \to \mathcal{T}_F$, $\mathcal{T}\tilde{R}(n,d) \cong \tilde{R}(n,d)_F \times \mathcal{T}_F$, which determines topological trivializations of the other bundles over $\mathcal{T}_F$. These trivializations give an explicit identification of fibers in the respective bundles over $\mathcal{T}_F$; in particular, we obtain homeomorphisms (again reusing the symbols h and $\tilde{h}$):

$$h: R(n,d)_F^{ss} \to R(n,d)_F^{ss}, \quad \tilde{h}: \tilde{R}(n,d)_F^{ss} \to \tilde{R}(n,d)_F^{ss}, \quad \tilde{h}: (\tilde{Z}_R^{ss})_F \to (\tilde{Z}_R^{ss})_F.$$

We will use the following notation for the various equivariant Lefschetz series of the above maps:

$$L_t^{GL(p)}(R(n,d)^{ss}) \equiv L_t^{GL(p)}(h: R(n,d)_F^{ss} \to R(n,d)_F^{ss})$$
$$L_t^{GL(p)}(\tilde{R}(n,d)^{ss}) \equiv L_t^{GL(p)}(\tilde{h}: \tilde{R}(n,d)_F^{ss} \to \tilde{R}(n,d)_F^{ss})$$
$$L_t^N(\tilde{Z}_R^{ss}) \equiv L_t^N(\tilde{h}: (\tilde{Z}_R^{ss})_F \to (\tilde{Z}_R^{ss})_F).$$

Also let:

$$L_t(\tilde{\mathcal{M}}(n,d)) \equiv L_t(\tilde{h}: \tilde{\mathcal{M}}(n,d)_F \to \tilde{\mathcal{M}}(n,d)_F)$$
$$IL_t(\mathcal{M}(n,d)) \equiv IL_t(h: \mathcal{M}(n,d)_F \to \mathcal{M}(n,d)_F).$$

Any of the actions induced by $h: F \to F$ as above will referred to as the "monodromy action". We obtain the following extension of Kirwan's formulas:

PROPOSITION 5.7. *The Lefschetz polynomial of the monodromy action on the blownup moduli space is given by* $L_t(\tilde{\mathcal{M}}(n,d)) = (1-t^2)L_t^{GL(p)}(\tilde{R}(n,d)^{ss})$ *where*

$$L_t^{GL(p)}(\tilde{R}(n,d)^{ss}) = L_t^{GL(p)}(R(n,d)^{ss}) - \sum_{1 \le k \le n^2} \sum_{R \in \mathcal{R}(k)} L_t^N(\tilde{Z}_R^{ss})$$
$$+ \sum_{1 \le k \le n^2} \sum_{R \in \mathcal{R}(k)} \sum_{\beta \in \mathcal{B}(R)} (-1)^{q(\beta)} w(\beta)^{-1} t^{2d(\beta)} (1-t^2)^{-1} (1-t^{2z(\beta)+2}) L_t^{N \cap \mathrm{Stab}(\beta)}(\tilde{Z}_R^{ss})$$

where N is the normalizer of R in $GL(p)$. Let N_0 be the identity component of N. For $R = \prod_{1 \le j \le s} GL(m_j)$, let V^* be the graded vector space

$$V^* = H^*(B(R \cap \mathrm{Stab}(\beta))) \otimes H^*(\tilde{Z}_R^{ss}//N_0)$$
$$= \bigotimes_{1 \le j \le s} H^*(B(GL(m_j) \cap \mathrm{Stab}(\beta))) \otimes H^*\left(\left[\prod_{1 \le j \le s} \tilde{\mathcal{M}}(n_j, d_j)\right]\right).$$

Then

$$L_t^{N\cap \mathrm{Stab}(\beta)}(\tilde{Z}_R^{ss}) = \sum_{i\geq 0}(-1)^i \operatorname{trace}(\operatorname{Inv}_H V^i)t^i$$

where $H = \pi_0(R(N\cap \mathrm{Stab}(\beta)))\subset \pi_0 N$ *acts by permuting factors and* trace $\operatorname{Inv}_H V^i$ *is the trace of the monodromy action on* $\operatorname{Inv}_H V^i$.

PROPOSITION 5.8. *Let* $\mathscr{R} = \bigcup_{1\leq k\leq n^2}\mathscr{R}(k)$. *For* $R = \prod_{1\leq j\leq s} GL(m_j)\in\mathscr{R}$ *define* $S(R,q,q',q'')$ *to be*

$$H^q\left(\left[\prod_{1\leq j\leq s}\tilde{\mathscr{M}}(n_j,d_j)\right]\right)\otimes H^{q'}\left(\prod_{1\leq j\leq s} BGL(m_j)\right)\otimes IH^m_{t(q'')}(\mathbb{P}(H^1(F,\mathrm{End}'_\oplus E))//R).$$

Then the intersection Lefschetz polynomial of the monodromy action on $\mathscr{M}(n,d)$ *is given by:*

$$IL_t(\mathscr{M}(n,d)) = L_t(\tilde{\mathscr{M}}(n,d)) - \sum_{i\geq 0}\sum_{R\in\mathscr{R}}\sum_{q+q'+q''=i}(-1)^i \operatorname{trace}\operatorname{Inv}_{\pi_0 N} S(R,q,q',q'')t^i$$

where N is the normalizer of R in GL(p) and trace $\operatorname{Inv}_{\pi_0 N} S(R,q,q',q'')$ *is the trace of the monodromy action on* $\operatorname{Inv}_{\pi_0 N} S(R,q,q',q'')$.

The proof of Propositions 5.7 and 5.8 proceeds in parallel with the Kirwan's proof ([16]) of Propositions 5.5 and 5.6. In addition, recall that Lefschetz series, in ordinary, equivariant, or intersection homology are multiplicative with respect to morphisms of homologically trivial fibrations, i.e. fibrations for which the associated spectral sequence degenerates at E^2 and that E^2 is a tensor product of the appropriate type of homology of the base with that of the fiber. Another consequence of multiplicativity is the following formula for $L_t([\prod_{1\leq j\leq s}\tilde{\mathscr{M}}(n_j,d_j)])$. Write $\{1,\ldots,s\}$ as the disjoint union of I_i $1\leq i\leq v$ so that for $i\in I_p$ and $j\in I_q$ $\tilde{\mathscr{M}}(n_i,d_i) = \tilde{\mathscr{M}}(n_j,d_j)$ if and only if $p=q$. Then

$$L_t\left(\left[\prod_{1\leq j\leq s}\tilde{\mathscr{M}}(n_j,d_j)\right]\right) = \prod_{1\leq p\leq v} L_t\left(\left[\prod_{j\in I_p}\tilde{\mathscr{M}}(n_j,d_j)\right]\right) \tag{5.9}$$

and, writing $r = \# I_p$ and $\tilde{\mathscr{M}} = \tilde{\mathscr{M}}(n_j,d_j)$ for $j\in I_p$, the Lefschetz polynomial in the blownup product, $L_t([\prod_{j\in I_p}\tilde{\mathscr{M}}(n_j,d_j)])$, is calculated by the inductive formula:

$$\begin{aligned} L_t\left(\left[\prod_{j\in I_p}\tilde{\mathscr{M}}(n_j,d_j)\right]\right) &= L_t\left(\left[\prod_{1\leq j\leq r}\tilde{\mathscr{M}}\right]\right) \\ &= (L_t(\tilde{\mathscr{M}}))^r + \sum_{2\leq k\leq r} L_t\left(\left[\prod_{k\leq j\leq r}\tilde{\mathscr{M}}\right]\right)(1-t^2)^{-1}(t^2 - t^{2(k-1)q}) \end{aligned} \tag{5.10}$$

where q is the complex dimension of $\tilde{\mathscr{M}}$ (compare [16, 3.10]).

(C) *Supertraces and Lefschetz series*

Let Σ_n be the symmetric group of permutations of $\{1,\ldots,n\}$ and I_n the set of partitions of n, i.e. sequences of integers $j_1\geq\ldots\geq j_q\geq 1$ such that $j_1+\cdots+j_q = n$. There is a bijection between I_n and a complete set $\{W_\pi \mid \pi\in I_n\}$ of complex irreducible Σ_n-modules. The W_π's are, in fact, defined over the rational numbers.

Let V be a finite dimensional super vector space defined over a field k of characteristic 0, i.e. V has $\mathbb{Z}/2$ grading $V = V_0\oplus V_1$. The n-fold tensor product $V^{\otimes n}$ is also a super vector space with $\mathbb{Z}/2$ grading

$$(V^{\otimes n})_j = \bigoplus_{i_1 + \cdots + i_n = j \bmod 2} V_{i_1} \otimes \cdots \otimes V_{i_n}, \quad j = 0, 1.$$

We now describe the Σ_n action on the n-fold tensor product $V^{\otimes n}$. If $\sigma \in \Sigma_n$ then

$$\sigma \cdot a_1 \otimes \cdots \otimes a_n = (-1)^p a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)} \tag{5.11}$$

where p is determined as follows. If σ is the transposition which interchanges $i < j$ then

$$p = (\deg a_{i+1} + \cdots + \deg a_{j-1})(\deg a_i + \deg a_j) + \deg a_i \deg a_j. \tag{5.12}$$

The sign for general σ is obtained by factoring σ into transpositions and repeatedly applying 5.12.

Given $\pi \in I_n$ define $V^\pi = W_\pi \otimes \mathrm{Hom}_{\Sigma_n}(W_\pi, V^{\otimes n})$; we obtain a natural direct sum decomposition of Σ_n-modules $V^{\otimes n} \cong \bigoplus_{\pi \in I_n} V^\pi$. V^π is a super vector space, the $\mathbb{Z}/2$ grading is given by $(V^\pi)_j = W_\pi \otimes \mathrm{Hom}_{\Sigma_n}(W_\pi, (V^{\otimes n})_j)$ $j = 0, 1$. A linear endomorphism $T: V \to V$ induces a linear map $T^\pi: V^\pi \to V^\pi$.

Recall that if $V' = V'_0 \oplus V'_1$ is a finite dimensional super vector space and $S: V' \to V'$ is a linear map then the *supertrace* of S is defined by $\mathrm{trace}_s(S) = \mathrm{trace}(\varepsilon S)$ where $\varepsilon: V' \to V'$ given by $\varepsilon(a_0 + a_1) = a_0 - a_1, a_j \in V'_j, j = 0, 1$. The k-algebra of endomorphisms of V, $\mathrm{End}(V)$, is also a super vector space with grading $\mathrm{End}(V) = \mathrm{End}(V)_0 \oplus \mathrm{End}(V)_1$ where $T \in \mathrm{End}(V)_j$ if and only if $T\varepsilon = (-1)^j \varepsilon T$.

Proposition 5.13. *There is a polynomial $P_\pi(x_1, \ldots, x_n)$ defined over the rationals such that for any finite dimensional super vector space V and endomorphism $T \in \mathrm{End}(V)_0$ the supertrace of T^π is* $\mathrm{trace}_s(T^\pi) = P_\pi(\mathrm{trace}_s(T), \mathrm{trace}_s(T^2), \ldots, \mathrm{trace}_s(T^n))$.

Proof. It is convenient to define $U^\pi = \mathrm{Hom}_{\Sigma_n}(W_\pi, V^{\otimes n})$. U^π is a super vector space with grading $(U^\pi)_j = \mathrm{Hom}_{\Sigma_n}(W_\pi, (V^{\otimes n})_j)$; furthermore, T induces an endomorphism $\bar{T}^\pi: U^\pi \to U^\pi$ of degree 0. Since $V^\pi = W_\pi \otimes U^\pi$ and $T^\pi = 1 \otimes \bar{T}^\pi$ we have $\mathrm{trace}_s(T^\pi) = (\dim W_\pi)\, \mathrm{trace}_s(\bar{T}^\pi)$ and so it suffices to prove the corresponding statement for $\mathrm{trace}_s(\bar{T}^\pi)$.

We first examine the case W_π is the trivial representation; it is associated to the partition $\pi = (n)$. $U^{(n)}$ is the super vector space of Σ_n invariant tensors.

The *r-th super symmetric power of* V, denoted $S^r(V)$, is defined to be the co-invariants of the Σ_r action on $V^{\otimes r}$, i.e. $S^r(V) = k \otimes_{\Sigma_r} V^{\otimes r}$. $S^r(V)$ inherits a super vector structure from $V^{\otimes r}$ and T induces an endomorphism $S^r(T)$: $S^r(V) \to S^r(V)$. Since the base field k has characteristic 0, there is a natural isomorphism $S^r(V) \cong U^{(r)}$ given by $1 \otimes_{\Sigma_r} x \mapsto \tau x$ where $\tau = \sum_{g \in \Sigma_r} g$. In particular, $\mathrm{trace}_s(S^r(T)) = \mathrm{trace}_s(\bar{T}^{(r)})$.

Write $V = V_0 \oplus V_1$ with $V_0 = k^m$ and $V_1 = k^p$. Without loss of generality, it can be assumed that T is given by a diagonal matrix with diagonal entries $(\lambda_1, \ldots, \lambda_m, \mu_1, \ldots, \mu_p)$. The following formula is a variant of André Weil's well known formula for zeta functions. There is an identity of formal power series:

$$\frac{1}{\det_s(1 - zT)} = \exp\left(\sum_{r \geq 1} \mathrm{trace}_s(T^r) \frac{z^r}{r} \right) \tag{5.14}$$

where

$$\det_s(1 - zT) = \frac{\prod_{j=1}^m (1 - z\lambda_j)}{\prod_{i=1}^p (1 - z\mu_i)}$$

is the *superdeterminant of* $1 - zT$. Using the formal identity $1/(1 - z\lambda_j) = \Sigma_{r \geq 0} z^r \lambda_j^r$ one observes that the coefficient of z^r on the left side of 5.14 is $\mathrm{trace}_s(S^r(T))$. The coefficient of z^r on the right side of 5.14 is clearly a polynomial in $\mathrm{trace}_s(T), \ldots, \mathrm{trace}_s(T^r)$ thus verifying the desired conclusion in the case W_π is the trivial representation.

The partitions of n are partially ordered by the relation $\rho = (\rho_1, \ldots, \rho_k) \geq \rho' = (\rho'_1, \ldots, \rho'_{k'})$ if for all j, $\sum_{i=1}^{j} \rho_i \geq \sum_{i=1}^{j} \rho'_i$ (see [12, Definition 3.2]). The irreducible Σ_n module corresponding to the partition $\pi = (n_1, \ldots, n_q)$ can be constructed as follows. Let $\Sigma_{n_1} \times \cdots \times \Sigma_{n_q}$ be the product of symmetric groups included into Σ_n in the obvious manner. Consider the induced Σ_n module $M = k(\Sigma_n/\Sigma_{n_1} \times \cdots \times \Sigma_{n_q})$. According to [12, Theorem 4.13], M is completely reducible with a decomposition of the form

$$M = W_\pi \oplus \bigoplus_{\rho > \pi} n_\rho W_\rho$$

where n_ρ are non-negative integers. We have verified the conclusion of the proposition in the case of the partition (n), corresponding to the trivial representation. Suppose the proposition has been verified for ρ with $\rho > \pi$. The decomposition of M yields

$$\mathrm{Hom}_{\Sigma_n}(M, V^{\otimes n}) = U^\pi \oplus \bigoplus_{\rho > \pi} n_\rho U^\rho, \quad \mathrm{Hom}_{\Sigma_n}(M, V^{\otimes n}) = U^{(n_1)} \otimes \cdots \otimes U^{(n_k)}.$$

T induces an endomorphism of $\mathrm{Hom}_{\Sigma_n}(M, V^{\otimes n})$ which we will denote by T_M. The supertrace of T_M is given by

$$\mathrm{trace}_s(T_M) = \mathrm{trace}_s(\bar{T}^{(n_1)} \otimes \cdots \otimes \bar{T}^{(n_q)}) = \prod_{j=1}^{q} \mathrm{trace}_s(\bar{T}^{(n_j)}).$$

We thus obtain the formula:

$$\mathrm{trace}_s(\bar{T}^\pi) = \prod_{j=1}^{q} \mathrm{trace}_s(\bar{T}^{(n_j)}) - \sum_{\rho > \pi} n_\rho \, \mathrm{trace}_s(\bar{T}^\rho).$$

By the inductive hypothesis the right side is a polynomial in $\mathrm{trace}_s(T), \ldots, \mathrm{trace}_s(T^n)$. $\square$

Remark. The proof of Proposition 5.13 gives an algorithm for the computation of the polynomials P_π.

Now suppose $V_* = \{V_i \mid i \geq 0\}$ is a graded vectôr space and $\phi: V_* \to V_*$ is a degree 0 endomorphism. Assume each V_i is finite dimensional. Recall the that *Lefschetz series* of ϕ is the formal power series defined by $L_t(\phi) = \Sigma_{i \geq 0}(-1)^i \,\mathrm{trace}(\phi_i) t^i$. For the moment assume that $V_i = 0$ for sufficiently large i so that $L_t(\phi)$ is a polynomial, the *Lefschetz polynomial* of ϕ. Let $V = \bigoplus_{i \geq 0} V_i$ which we regard as a super vector with grading $V = V^0 \oplus V^1$ where $V^0 = \bigoplus_{k \geq 0} V_{2k}$ and $V^1 = \bigoplus_{k \geq 0} V_{2k+1}$. For a scalar t define an endomorphism $\phi_t: V \to V$ by $\phi_t = \bigoplus_{i \geq 0} t^i \phi_i$. Observe that $\mathrm{trace}_s(\phi_t) = L_t(\phi)$ and $\mathrm{trace}_s((\phi_t)^k) = L_{t^k}(\phi^k)$. Σ_n acts on the graded tensor product $V_*^{\otimes n}$ via the formulas 5.11 and 5.12 and there is a direct sum decomposition $V_*^{\otimes n} \cong \bigoplus_{\pi \in I_n} V_*^\pi$; moreover, ϕ induces an endomorphism $\phi^\pi: V_*^\pi \to V_*^\pi$.

Proposition 5.15. *There is a polynomial $P_\pi(x_1, \ldots, x_n)$ defined over the rational numbers such that for any finite dimensional graded vector space V_* and degree* 0 *endomorphism* $\phi: V_* \to V_*$, $L_t(\phi^\pi) = P_\pi(L_t(\phi), L_{t^2}(\phi^2), \ldots, L_{t^n}(\phi^n))$.

Proof. Apply Lemma 5.13 to $(\phi^\pi)_t = (\phi_t)^\pi$. $\square$

By applying truncation operators to V_*, the same result is seen to also apply to Lefschetz series. To illustrate Proposition 5.15 consider the case $n = 2$. The graded vector space $V_*^{\otimes 2}$ decomposes as a direct sum $V_*^{\otimes 2} = V_*^s \oplus V_*^a$ of the symmetric tensors V_*^s and the antisymmetric tensors V_*^a. An easy calculation reveals:

$$L_t(\phi^s) + L_t(\phi^a) = L_t(\phi^{\otimes 2}) = [L_t(\phi)]^2,$$
$$L_t(\phi^s) - L_t(\phi^a) = L_{t^2}(\phi^2).$$

This proves:

PROPOSITION 5.16. *If* $n = 2$ *the polynomials of Proposition* 5.15 *are given by* $P_s(x_1, x_2) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2$ *and* $P_a(x_1, x_2) = \frac{1}{2}x_1^2 - \frac{1}{2}x_2$. *Thus*

$$L_t(\phi^s) = \tfrac{1}{2}[L_t(\phi)]^2 + \tfrac{1}{2}L_{t^2}(\phi^2)$$
$$L_t(\phi^a) = \tfrac{1}{2}[L_t(\phi)]^2 - \tfrac{1}{2}L_{t^2}(\phi^2).$$

Suppose H is a subgroup of Σ_n and M_* is a graded H-module such that each M_i is finite dimensional. H acts on $V_*^{\otimes n}$ by restriction of the Σ_n action. Let $\mathrm{Inv}_H(M_* \otimes V_*^{\otimes n})$ denote the invariant part of the H-action on the tensor product $M \otimes V^{\otimes n}$. A graded endomorphism $\phi\colon V \to V$ of degree 0 induces an endomorphism $\phi^H\colon \mathrm{Inv}_H(M_* \otimes V_*^{\otimes n}) \to \mathrm{Inv}_H(M_* \otimes V_*^{\otimes n})$. For $\pi \in I_n$ and $j \geq 0$ let $m(M_j, \pi)_H$ be the multiplicity of the trivial representation in the decomposition of $M_j \otimes W_\pi$ as direct sum of irreducible complex H-modules. Define $m_t(M_*, \pi)_H = \Sigma_{j \geq 0}(-1)^j m(M_j, \pi)_H t^j$.

THEOREM 5.17. *For any graded vector space* V_* *of finite type and for any degree* 0 *endomorphism* $\phi\colon V_* \to V_*$:

$$L_t(\phi^H) = \sum_{\pi \in I_n} (\dim W_\pi)^{-1} m_t(M_*, \pi)_H P_\pi(L_t(\phi), L_{t^2}(\phi^2), \ldots, L_{t^n}(\phi^n))$$

where the polynomials P_π *are given by Proposition* 5.15.

Proof. Let $U_*^\pi = \mathrm{Hom}_{\Sigma_n}(W_\pi, V_*^{\otimes n})$. ϕ induces an endomorphism $\bar{\phi}^\pi\colon U_*^\pi \to U_*^\pi$ such that $\phi^\pi = 1 \otimes \bar{\phi}^\pi$. Hence by Proposition 5.15

$$L_t(\bar{\phi}^\pi) = (\dim W_\pi)^{-1} L_t(\phi^\pi) = (\dim W_\pi)^{-1} P_\pi(L_t(\phi), L_{t^2}(\phi^2), \ldots, L_{t^n}(\phi^n)).$$

Let K_* be a graded vector space such that $\dim(K_j) = m(M_j, \pi)_H$. Note that $L_t(1_{K_*}) = m_t(M_*, \pi)_H$ where 1_{K^*} is the identity map. We have that

$$\mathrm{Inv}_H(M_* \otimes V_*^{\otimes n}) = \bigoplus_{\pi \in I_n} K_* \otimes U_*^\pi$$

and so $L_t(\phi^H) = \sum_{\pi \in I_n} L_t(1_{K^*}) L_t(\bar{\phi}^\pi)$. $\square$

(D) Calculation of $\lambda_{n,d}$ and $p_{n,d}(t)$

Let (n, d) be a pair of integers with $n \geq 1$. Let K be a fibered knot in a rational homology 3-sphere M with a Seifert surface S (which is a fiber) of genus g and monodromy action $f\colon S \to S$. The (un-normalized) Alexander polynomial of K will be denoted by $c(t)$. The *normalized* Alexander polynomial of K is defined by $\tilde{c}(t) = t^{-g}c(t)$. Let F be the closed surface obtained by "capping off" S via longitudinal surgery on K. The monodromy action gives rise to an action on the moduli space $\mathcal{M}(n, d)$ over F as described in §5(B). This action will also be denoted by f.

Recall that $\mathcal{M}(n, d)$ is a fiber bundle over the Jacobian of F:

$$\mathcal{M}_{\det}(n,d) \to \mathcal{M}(n,d) \to J(F)$$

where $\mathcal{M}_{\det}(n, d)$ is the subvariety consisting of bundles with fixed determinant. By [9, Theorem I], the intersection Lefschetz number of the action f on $\mathcal{M}_{\det}(n, d)$ is equal to the IH-intersection number of the middle dimensional, middle perversity homology classes which are determined (in the manner described in [9]) by the graph of f: $\mathcal{M}_{\det}(n, d) \to \mathcal{M}_{\det}(n, d)$ and the diagonal $\Delta \subset \mathcal{M}_{\det}(n, d) \times \mathcal{M}_{\det}(n, d)$. The latter intersection number is easily seen to coincide with the intersection number $\lambda_{n,d}$ which was defined in §3.

Since the above fibering is homologically trivial (see [10]), the multiplicativity of Lefschetz polynomials implies that

$$IL_t(\mathcal{M}(n,d)) = L_t(J(F))\, IL_t(\mathcal{M}_{\det}(n,d)) = c(t)\, IL_t(\mathcal{M}_{\det}(n,d)).$$

The following proposition is a summary of Frohman's calculation in [4, §3].

Proposition 5.18.

(1) *There are polynomials $P(x_1, \ldots, x_n)$ and $q(t)$ depending only on (n, d) and g such that*

$$(1 - t^2)\, L_t^{GL(p)}(R(n,d)^{ss}) = \frac{P(c(t), c(t^3), \ldots, c(t^{2n-1}))}{q(t)}.$$

(2) *The value of $(1 - t^2)\, L_t^{GL(p)}(R(n,d)^{ss})$ at $t = 1$ is a polynomial, depending only on (n, d), in the derivatives of order $2k$, $0 \le k \le n - 1$, of the normalized Alexander polynomial $\tilde{c}(t)$.* □

Remark. The calculation carried out in [4], for fibered knots employed an extension of the methods of [2]. This involves an equivalent description of $R(n, d)^{ss}//GL(p)$ in terms of holomorphic connections. There is an alternative and entirely parallel method of performing the same computation which uses the geometric invariant theory description of the relevant spaces as given in [18] where an algorithm for the computation of the equivariant Poincaré series of $R(n, d)^{ss}$ is detailed. The calculation of [4] (which deals with the case of a fibered knot) can be extended without serious difficulty to the case of the action of a correspondence arising from an arbitrary homologically trivial knot; formally, at least, the computation of the various Lefschetz series is the same. The blowup process involved in the computation of $IL_t(\mathcal{M}(n, d))$ complicates the treatment of the case of the action of a general correspondence via the method of §5(B). For this reason, we have restricted our attention to the case of fibered knots where the technique of working over Teichmüller space overcomes this complication. We conjecture that Theorems 5.21, 5.22, and 6.4 remain valid in the non-fibered case.

Let $\hat{M}_q$ be the q-th branched cyclic cover of M along K. If $\phi_{n',d'}$ is the monodromy action on $IH_*^m(\mathcal{M}(n', d'))$ associated to $K \subset M$ (and the Seifert surface S) then $IL_t(\mathcal{M}(n', d'); \hat{M}_q) = L_{t^q}(\phi^q_{n',d'})$. In particular, if $c_q(t)$ is the Alexander polynomial of $K \subset \hat{M}_q$ it is straightforward to show that

$$c_q(t) = L_{t^q}(\phi^q_{1,0}) = \prod_{i=0}^{q-1} c(\omega_q^i t) \tag{5.19}$$

where $\omega_q = e^{2\pi i/q}$ is a primitive q-th root of unity (the second equality utilizes the formula $\det(A^q - t^q) = \prod_{i=0}^{q-1} \det(A - \omega_q^i t)$ for a square matrix A).

Remark. Note that $\hat{M}_q$ need not be a rational homology sphere. Our computation, in the case of a fibered knot, is still valid as we have presented it even if M has positive first Betti number; however, in this situation $\lambda_{n,d} = 0$ (this is clear when $n = 1$; for $n > 1$, it may be proved by induction on n).

In [4] an algorithm is given for computing the polynomials which appear on the right in Proposition 5.18(1). Combining this with 5.7, 5.8, 5.9, 5.10, 5.17, and 5.19 we obtain an inductive algorithm for the calculation of $IL_t(\mathcal{M}(n,d)) = c(t)\, IL_t(\mathcal{M}_{\det}(n,d))$ in terms of $c(\omega_i^j t^{2k-1})$ where $1 \leq i \leq n$, $0 \leq j \leq i-1$, and $1 \leq k \leq n$. In summary, we have:

Proposition 5.20. *Let X be the set of variables*

$$X = \{x_{ijk} \mid 1 \leq i \leq n, 0 \leq j \leq i-1, 1 \leq k \leq n, \textit{if } k = n \textit{ then } i = 1\}.$$

(1) *There are polynomials $P(X)$ and $q(t)$ depending only on (n, d) and g (and its parity) such that*

$$IL_t(\mathcal{M}_{\det}(n,d)) = \frac{P(\{c(\omega_i^j t^{2k-1})\})}{q(t)}.$$

(2) *There is a polynomial $T(X)$, depending on (n, d) and a priori on g (and its parity) such that the value of $IL_t(\mathcal{M}_{\det}(n,d))$ at $t = 1$ is $T(\{\tilde{c}^{(2k-2)}(\omega_i^j)\})$ where $\tilde{c}^{(r)}$ is the r-th derivative of the normalized Alexander polynomial.* □

In §6 we compute T in the case $n = 2$ and d is even; see Theorem 6.4. Theorem 1.3 implies that for all (n, d), the polynomial T is independent of the genus g. Although an alternative algebraic proof of this fact (by induction on n) is probably possible, Theorem 1.3 provides a cogent geometric explanation. We have proved:

Theorem 5.21 (Calculation of the invariant $\lambda_{n,d}$). *Let X be the set of variables $X = \{x_{ijk} \mid 1 \leq i \leq n,\ 0 \leq j \leq i-1,\ 1 \leq k \leq n,\ \textit{if } k = n \textit{ then } i = 1\}$. There is a polynomial $T(X)$, depending only on (n, d) such that $\lambda_{n,d} = T(\{\tilde{c}^{(2k-2)}(\omega_i^j)\})$ where $\tilde{c}^{(r)}$ is the r-th derivative of the normalized Alexander polynomial.* □

We may also apply Proposition 5.20 to the computation of the polynomial invariants $p_{n,d}(t)$. Substituting the Taylor expansion of $c(\omega_i^j t^{2k-1})$ in t at $t = 1$ in the expression for $IL_t(\mathcal{M}_{\det}(n,d))$ given by Proposition 5.20(1) and applying Theorem 1.3, we conclude:

Theorem 5.22. *There are polynomials depending only on (n, d) whose evaluation at $\{\tilde{c}^{(2k-2)}(\omega_i^j)\}$ (the set of values of the normalized Alexander polynomial of K and its derivatives at m-th roots of unity, $m \leq n$) yield the coefficients of $p_{n,d}(t)$.*

§6. AN EXPLICIT FORMULA FOR $\lambda_{2,0}$

In this section we will focus our attention on the moduli space of rank 2 semistable bundles of even degree d and compute the numerical invariant $\lambda_{2,0}$ for a fibered knot; see Theorem 6.4. Our treatment will closely parallel the computations of Kirwan in [16, §4–5]. Let $c(t)$ denote the (un-normalized) Alexander polynomial of the knot K. Our first task is to compute the Lefschetz polynomial of the monodromy action on the blownup object $\tilde{\mathcal{M}}(2,d)$. The Lefschetz polynomial $L_t(\tilde{\mathcal{M}}(2,d))$ is given by

$$L_t(\tilde{\mathcal{M}}(2,d)) = (1-t^2)\, L_t^{GL(p)}(\tilde{R}(2,d)^{ss})$$

and so we turn our attention to the calculation of $L_t^{GL(p)}(\tilde{R}(2,d)^{ss})$. The equivariant

Lefschetz series, prior to the blowup process, is given by:

$$c(t)^{-1}(1-t^2)L_t^{GL(p)}(R(2,d)^{ss}) = (1-t^2)^{-1}(1-t^4)^{-1}(c(t^3) - t^{2g+2}c(t)).$$

This calculation is done in [4].

To obtain $R_1(2,d)^{ss}$ we blowup $R(2,d)^{ss}$ along the stratum $GL(p)Z^{ss}_{GL(2)}$ corresponding to holomorphic bundles E which split as the direct sum of two isomorphic line bundles, $E = L \oplus L$. In particular, this blowup contributes a term

$$L_t^{GL(p)}(GL(p)Z^{ss}_{GL(2)})(1-t^2)^{-1}(t^2 - t^{6g})$$

which must be added to $L_t^{GL(p)}(R(2,d)^{ss})$. Now

$$\begin{aligned} L_t^{GL(p)}(GL(p)Z^{ss}_{GL(2)}) &= L_t^{N(GL(2))}(GL(p)Z^{ss}_{GL(2)}) \\ &= P_t(BGL(2))\,L_t(\tilde{\mathscr{M}}(1,\tfrac{1}{2}d)) \\ &= (1-t^2)^{-1}(1-t^4)^{-1}c(t). \end{aligned}$$

To obtain $L_t^{GL(p)}(R_1(2,d)^{ss})$ we must subtract terms of the form $t^{2\,\operatorname{codim} S_\beta}L_t^{GL(p)}(S_\beta)$ for each unstable stratum S_β. In this case there is only one such stratum which has complex codimension $2g-1$.

$$\begin{aligned} L_t^{GL(p)}(S_\beta) &= L_t^{(T\times GL(p/2))/\mathbb{C}^*}(Z_\beta) \\ &= (1-t^2)^{-1}(1-t^{2g})\,L_t(\tilde{\mathscr{M}}(1,\tfrac{1}{2}d))\,P_t(BT) \\ &= (1-t^2)^{-1}(1-t^{2g})c(t)(1-t^2)^{-2} \end{aligned}$$

It follows that

$$\begin{aligned} c(t)^{-1}(1-t^2)L_t^{GL(p)}(R_1(2,d)^{ss}) &= (1-t^2)^{-1}(1-t^4)^{-1}(c(t^3) - t^{2g+2}c(t)) \\ &\quad + (1-t^4)^{-1}((1-t^2)^{-1}(t^2 - t^{6g})) \\ &\quad - (1-t^2)^{-1}(t^{4g-2}(1-t^2)^{-1}(1-t^{2g})). \end{aligned}$$

$\tilde{R}(2,d)$ is obtained from $R_1(2,d)$ by blowing up the stratum corresponding to holomorphic bundles E of the form $E = L_1 \oplus L_2$ where the line bundles L_1 and L_2 are not isomorphic. Thus when we blowup $R_1(2,d)^{ss}$ along $GL(p)\tilde{Z}^{ss}_T$ we must add

$$L_t^{N(T)}(\tilde{Z}^{ss}_T)(1-t^2)^{-1}(t^2 - t^{4g-4})$$

to the equivariant Lefschetz series. The cohomology $H^*(\tilde{Z}^{ss}_T/GL(p/2))$ has a $\Sigma_2 = \mathbb{Z}/2$ action which arises from permutation of the line bundle summands in the decomposition $E = L_1 \oplus L_2$. To compute $L_t^{N(T)}(\tilde{Z}^{ss}_T)$ we must examine how the monodromy action acts on the invariant part of this cohomology. Proceeding as in the proof of [16, (4.8)] we obtain:

$$L_t^{N(T)}(\tilde{Z}^{ss}_T) = \tfrac{1}{2}(1+t^2)c(t) + \tfrac{1}{2}(1-t^2)c(-t) + (1-t^2)^{-1}(t^2 - t^{2g})$$

Finally, $L_t^{GL(p)}(\tilde{R}(2,d)^{ss})$ is obtained by subtracting terms of the form $t^{2\,\operatorname{codim} S_\beta}L_t^{GL(p)}(S_\beta)$ for each unstable stratum S_β in the second blowup. Again, there is only one such stratum. Its complex codimension is $g-1$ and its equivariant Lefschetz series is given by

$$\begin{aligned} L_t^{GL(p)}(S_\beta) &= (1-t^2)^{-1}(1-t^{2g-2})\,L_t(\tilde{Z}^{ss}_T/GL(p/2))\,P_t(BT) \\ &= (1-t^2)^{-1}(1-t^{2g-2})c(t)(c(t) + (1-t^2)^{-1}(t^2 - t^{2g}))(1-t^2)^{-2} \end{aligned}$$

This proves:

PROPOSITION 6.1. *The Lefschetz polynomial of the monodromy action on $\tilde{\mathcal{M}}(2,d)$ associated to a fibered knot with Alexander polynomial $c(t)$ is given by*

$$\begin{aligned}
c(t)^{-1} L_t(\tilde{\mathcal{M}}(2,d)) = L_t(\tilde{\mathcal{M}}_{\det}(2,d)) &= (1-t^2)^{-1}(1-t^4)^{-1}(c(t^3) - t^{2g+2}c(t)) \\
&\quad + (1-t^4)^{-1}((1-t^2)^{-1}(t^2 - t^{6g})) \\
&\quad - (1-t^2)^{-1}(t^{4g-2}(1-t^2)^{-1}(1-t^{2g})) \\
&\quad + (1-t^4)^{-1}(\tfrac{1}{2}(1+t^2)c(t) + \tfrac{1}{2}(1-t^2)c(-t) \\
&\qquad + (1-t^2)^{-1}(t^2 - t^{2g}))(1-t^2)^{-1}(t^2 - t^{4g-4}) \\
&\quad - (1-t^2)^{-1}(t^{2g-2}(c(t) \\
&\qquad + (1-t^2)^{-1}(t^2 - t^{2g}))(1-t^2)^{-1}(1-t^{2g-2})).
\end{aligned}$$

The Lefschetz number the monodromy action is computed by evaluating $L_t(\tilde{\mathcal{M}}_{\det}(2,d))$ at $t = 1$ using L'Hôpital's rule, yielding:

$$\tfrac{1}{4}((2g^3 - 2g^2) + (g^2 - 10g + 3)c(1) + (2g-3)c(-1) - (g-6)c'(1) + 2c''(1)).$$

Since $c(t) = t^{2g}c(1/t)$ we have $c'(1) = gc(1)$.

COROLLARY 6.2. *The Lefschetz number of the monodromy action on the blownup object $\tilde{\mathcal{M}}_{\det}(2,d)$ is:*

$$\tfrac{1}{4}((2g^3 - 2g^2) + (-4g+3)c(1) + (2g-3)c(-1)) + \tfrac{1}{2}c''(1).$$

Our goal is to compute the intersection homology Lefschetz polynomial, $IL_t(\mathcal{M}(2,d))$. The relationship between the Lefschetz polynomial in $\tilde{\mathcal{M}}(2,d)$ and the intersection Lefschetz polynomial in $\mathcal{M}(2,d)$ is given by (see Proposition 5.8 or compare [16, §5]):

$$\begin{aligned}
IL_t(\mathcal{M}(2,d)) &= L_t(\tilde{\mathcal{M}}(2,d)) - c(t)\sum_{q\ge 0}(-1)^q t^q \dim IH_{t(q)}(\mathbb{P}(H^1(M,\mathrm{End}'_{\oplus}E))//SL(2)) \\
&\quad - \sum_{i\ge 0}(-1)^i t^i \sum_{q+q'=i} \operatorname{trace}\operatorname{Inv}_{\pi_0(N)} H^q(J\tilde{\times}J)\otimes IH_{t(q')}(\mathbb{P}(H^1(M,\mathrm{End}'_{\oplus}E))//T).
\end{aligned}$$

Here $J = \mathcal{M}(1,\frac{1}{2}d)$ is the Jacobian, $J\tilde{\times}J$ is the product blownup along the diagonal and N is the normalizer of T, the maximal torus of $SL(2)$. Recall that the integer function $t(s)$ is defined by

$$t(s) = \begin{cases} s-2 & \text{if } s < \dim\mathbb{P}(H^1(M,\ \mathrm{End}'_{\oplus}E))//R \\ s & \text{otherwise} \end{cases}$$

where in the formula above $R = SL(2)$ or T. Let $[x]$ denote the integer part of $x \ge 0$. The intersection Poincaré polynomial

$$IP_t(\mathbb{P}(H^1(M,\mathrm{End}'_{\oplus}E))//SL(2)) = \sum_{j=0}^{6g-8}(-1)^j b_j t^j$$

is computed in [16, §5]: $b_j = 0$ for j odd and

$$b_{2j} = \begin{cases} [\frac{1}{2}j] + 1 & \text{if } 0 \le j \le g-2 \\ [\frac{1}{2}g] & \text{if } g-1 \le j \le [3g/2]-2. \end{cases}$$

Let $\Phi(t) = \sum_{q\ge 0}(-1)^q t^q \dim IH_{t(q)}(\mathbb{P}(H^1(M,\mathrm{End}'_{\oplus}E))//SL(2))$. The coefficients of $\Phi(t)$ are obviously determined by those of the above intersection Poincaré polynomial. We will need to know the value of $\Phi(t)$ at $t = 1$. An easy calculation yields:

$$\Phi(1) = g^2 + (-\tfrac{5}{4} + \tfrac{1}{4}(-1)^g)g + \tfrac{1}{4} - \tfrac{1}{4}(-1)^g.$$

Kirwan shows [16, p. 262] that the dimension of the $+1$ and -1 eigenspaces of the action of $\pi_0 N = \mathbb{Z}/2$ on $IH_{2j}(\mathbb{P}(H^1(M, \operatorname{End}'_{\oplus} E))//T)$ are, respectively, $[\frac{1}{2}\min(j+2, 2g-2-j)]$ and $[\frac{1}{2}\min(j+1, 2g-3-j)]$. Define polynomials

$$U^+(t) = \sum_{j=1}^{2g-4} [\tfrac{1}{2}\min(j+1, 2g-2-j)]\, t^{2j}$$

$$U^-(t) = \sum_{j=2}^{2g-5} [\tfrac{1}{2}\min(j, 2g-3-j)]\, t^{2j}.$$

We will need the values of these polynomials at $t = 1$:

$$U^+(1) = \tfrac{1}{2}g^2 - g + \tfrac{1}{4} - \tfrac{1}{4}(-1)^g$$

$$U^-(1) = \tfrac{1}{2}g^2 - 2g + \tfrac{7}{4} + \tfrac{1}{4}(-1)^g.$$

The Lefschetz polynomials of the blownup product $J \tilde{\times} J$ and its quotient $J \tilde{\times} J/\pi_0 N$ are given by

$$L_t(J \tilde{\times} J) = c(t)^2 + c(t)(1-t^2)^{-1}(t^2 - t^{2g})$$

$$L_t(J \tilde{\times} J/\pi_0 N) = \tfrac{1}{2}c(t)^2 + \tfrac{1}{2}c(t)c(-t) + c(t)(1-t^2)^{-1}(t^2 - t^{2g}).$$

Note that $H^*(J \tilde{\times} J/\pi_0 N)$ can be identified with the $+1$ eigenspace of the action of $\pi_0 N$ on $H^*(J \tilde{\times} J)$. Thus the Lefschetz polynomial

$$\sum_{i \geq 0} (-1)^i t^i \sum_{q+q'=i} \operatorname{trace} \operatorname{Inv}_{\pi_0(N)} H^q(J \tilde{\times} J) \otimes IH_{t(q')}(\mathbb{P}(H^1(M, \operatorname{End}'_{\oplus} E))//T)$$

is given by

$$\{\tfrac{1}{2}c(t)^2 + \tfrac{1}{2}c(t)c(-t) + c(t)(1-t^2)^{-1}(t^2 - t^{2g})\}\, U^+(t) + \{\tfrac{1}{2}c(t)^2 - \tfrac{1}{2}c(t)c(-t)\}\, U^-(t).$$

This proves

PROPOSITION 6.3. *The intersection Lefschetz polynomial of the monodromy action on $\mathcal{M}(2, d)$ associated to a fibered knot with Alexander polynomial $c(t)$ is given by*

$$c(t)^{-1} IL_t(\mathcal{M}(2,d)) = IL_t(\mathcal{M}_{\det}(2,d)) = L_t(\tilde{\mathcal{M}}_{\det}(2,d)) - \Phi(t)$$
$$- \{\tfrac{1}{2}c(t) + \tfrac{1}{2}c(-t) + (1-t^2)^{-1}(t^2 - t^{2g})\}\, U^+(t)$$
$$- \{\tfrac{1}{2}c(t) - \tfrac{1}{2}c(-t)\}\, U^-(t).$$

The intersection Lefschetz number is obtained by evaluating the above polynomial at $t = 1$. This yields:

$$\tfrac{1}{4}((2g^3 - 2g^2) + (-4g+3)c(1) + (2g-3)c(-1)) + \tfrac{1}{2}c''(1) - (g^2 + (-\tfrac{5}{4} + \tfrac{1}{4}(-1)^g)g$$
$$+ \tfrac{1}{4} - \tfrac{1}{4}(-1)^g) - \{\tfrac{1}{2}c(1) + \tfrac{1}{2}c(-1) + g - 1\}(\tfrac{1}{2}g^2 - g + \tfrac{1}{4} - \tfrac{1}{4}(-1)^g)$$
$$- \{\tfrac{1}{2}c(1) - \tfrac{1}{2}c(-1)\}(\tfrac{1}{2}g^2 - 2g + \tfrac{7}{4} + \tfrac{1}{4}(-1)^g)$$

which simplifies to

$$-\tfrac{1}{2}g(g-1)c(1) + \tfrac{1}{4}\{(-1)^g c(-1) - c(1)\} + \tfrac{1}{2}c''(1).$$

The normalized Alexander polynomial, $\tilde{c}(t)$, is defined by $\tilde{c}(t) = t^{-g}c(t)$. Taking derivatives have $c'(t) = gt^{g-1}\tilde{c}(t) + t^g\tilde{c}'(t)$ and $c''(t) = g(g-1)t^{g-2}\tilde{c}(t) + 2gt^{g-1}\tilde{c}'(t) + t^g\tilde{c}''(t)$

and so $c(1) = \tilde{c}(1)$, $c(-1) = (-1)^g \tilde{c}(-1)$, and $c''(1) = \tilde{c}''(1) + g(g-1)\tilde{c}(1)$. Substituting into the above expression yields:

THEOREM 6.4. *The intersection Lefschetz number of the monodromy action on $\mathcal{M}_{\det}(2, d)$ is:*

$$\lambda_{2,0} = \tfrac{1}{4}\{\tilde{c}(-1) - \tilde{c}(1)\} + \tfrac{1}{2}\tilde{c}''(1). \qquad \square$$

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